Graded Computation Tree Logic

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In modal logics, *graded (world) modalities* have been deeply investigated as a useful framework for generalizing standard existential and universal modalities in such a way that they can express statements about a given number of immediately accessible worlds. These modalities have been recently investigated with respect to the μCALCULUS, which have provided succinctness, without affecting the satisfiability of the extended logic, that is, it remains solvable in EXPTIME. A natural question that arises is how logics that allow reasoning about paths could be affected by considering *graded path modalities*. In this article, we investigate this question in the case of the branching-time temporal logic CTL (GCTL, for short). We prove that, although GCTL is more expressive than CTL, the satisfiability problem for GCTL remains solvable in EXPTIME, even in the case that the graded numbers are coded in binary. This result is obtained by exploiting an automata-theoretic approach, which involves a model of alternating automata with satellites. The satisfiability result turns out to be even more interesting as we show that GCTL is at least exponentially more succinct than graded μCALCULUS.

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1. INTRODUCTION

Temporal logics are a special kind of *modal logics* that provide a formal framework for qualitatively describing and reasoning about how the truth values of assertions change over time. First pointed out by Pnueli [1977], these logics turn out to be particularly suitable for reasoning about correctness of concurrent programs [Pnueli 1981].

Depending on the view of the underlying nature of time, two types of temporal logics are mainly considered [Lamport 1980]. In *linear-time temporal logics*, such as LTL [Pnueli 1977], time is treated as if each moment in time has a unique possible future. Conversely, in *branching-time temporal logics*, such as CTL [Clarke and Emerson 1981] and CTL* [Emerson and Halpern 1986], each moment in time may split into various possible futures and *existential* and *universal quantifiers* are used to express properties along one or all the possible futures. In modal logics, such as ALC [Schmidt-Schauß

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and Smolka 1991] and μCALCULUS [Kozen 1983], these kinds of quantifiers have been generalized by means of *graded (worlds) modalities* [Fine 1972; Tobies 2001], which allow to express properties such as "there exist at least *n* accessible worlds satisfying a certain formula" or "all but *n* accessible worlds satisfy a certain formula." For example, in a multitasking scheduling specification, we can express properties such as every time a computation is invoked, immediately next there are at least two empty records in the task allocation table available for the allocation of two tasks that take care of the computation, without expressing exactly which records they are. This generalization has been proved to be very powerful, as it allows to express system specifications in a very succinct way. In some cases, the extension makes the logic much more complex. An example is the guarded fragment of the first order logic, which becomes undecidable when extended with a very weak form of counting quantifiers [Grädel 1999]. In some other cases, one can extend a logic with very strong forms of counting quantifiers without increasing the computational complexity of the obtained logic. For example, this is the case for μALCQ (see Baader et al. [2003] for a recent handbook) and GμCALCULUS [Kupferman et al. 2002; Bonatti et al. 2008], for which the decidability problem is EXPTIME-COMPLETE.

Despite its high expressive power, the μCALCULUS is considered in some sense a low-level logic, making it "unfriendly" for users, whereas simpler logics, such as CTL, can naturally express complex properties of computation trees. Therefore, an interesting and natural question that arises is how the extension of CTL with graded modalities can affect its expressiveness and decidability. There is a technical challenge involved in such an extension, which makes this task nontrivial. In the μCALCULUS, and other modal logics studied in the graded context so far, the existential and universal quantifiers range over the set of successors, thus it is easy to count the domain and its elements. In CTL, on the other hand, the underlying objects are both states and paths. Thus, the concept of graded must relapse on both of them. We solve this problem by introducing *graded path modalities* that extend to classes of paths the generalization induced to successor worlds by classical graded modalities, that is, they allow to express properties such as "there are at least *n* classes of paths satisfying a formula" and "all but at most less than *n* classes of paths satisfy a formula." We call the logic CTL extended with graded path modalities GCTL, for short.

A point that requires few considerations here is how we count paths along the model. We address this question by embedding in our framework a generic equivalence relation on the set of paths, but satisfying specific consistency properties. Therefore, the decisional algorithm we propose is very general and can be applied to different definitions of GCTL, along with different ways to identify the classes of paths. Along this line, one can observe that a state in a model can have only one direct successor, but possibly different paths going through it. This must be taken into account while satisfying a given graded path property. To deal this difficulty, we introduce a combinatorial tool which applies to a wide class of interesting equivalences. The tool is the partitioning of a natural number, that is, we consider all possible decompositions of a number into its summands (e.g., $3 = 3 + 0 = 2 + 1 = 1 + 1 + 1$). This is used to distribute a set of different paths emerging from a state onto all its direct successors. Note that, while CTL linearly translates to μCALCULUS, this complication makes the translation of GCTL to GμCALCULUS not easy at all. Indeed, we show such a translation with a double-exponential blow-up, by taking into account the path partitioning.

As a special equivalence class over paths, we also consider that one induced by the minimality and conservativeness requirements along the paths [Mogavero 2007; Bianco et al. 2009, 2010]. The minimality property allows to decide GCTL formulas on a restricted but significant space domain, that is, the set of paths of interest, in a very natural way. In more detail, it is enough to consider only the part of a system behavior that is effectively responsible for the satisfiability of a given formula, whenever each of its extensions satisfies the formula as well. So, we only take into account a set of noncomparable paths satisfying the same property using in practice a particular equivalence relation on the set of all paths. Moreover, if we drop the minimality, it may happen that to discuss the existence of a path in a structure does not make sense anymore. This is the case, for example, when the existence of a nonminimal path satisfying a formula may induce also the existence of an infinite number of paths satisfying it.

The ability of GCTL to reason about numbers of paths turns out to be suitable in several contexts. For example, it can be useful to query XML documents [Arenas et al. 2007; Libkin and Sirangelo 2008]. These documents, indeed, can be viewed as labeled unranked trees [Barceló and Libkin 2005] and GCTL allows reasoning about a number of links among tags of descendant nodes, in a very succinct way, without naming any of the intermediate ones. We also note that our framework of graded path quantifiers has some similarity with the concept of *cyclomatic complexity*, as it was defined by McCabe [1976] in a seminal work in software engineering. McCabe studied a way to measure the complexity of a program, identifying it in the number of independent instruction flows. From an intuitive point of view, since graded path quantifiers allow to specify how many classes of computational paths satisfying a given property reside in a program, GCTL subsumes the cyclomatic complexity, where the independence concept can be embedded into an apposite equivalence class. As another and more practical example of an application of GCTL, consider again the previous multitasking scheduling, where we may want to check that every time a nonatomic (i.e., non-one-step) computation is required, then there are at least n distinct (i.e., non–completely equivalent) computational flows that can be executed. This property can be easily expressed in GCTL. There are also several other practical examples that show the usefulness of GCTL and we refer to Ferrante et al. [2008, 2009] for a list of them.

The introduced framework of graded path modalities turns out to be very efficient in terms of expressiveness and complexity. Indeed, we prove that GCTL is more expressive than CTL, it retains the tree and the finite model properties, and its satisfiability problem is solvable in EXPTIME, therefore not harder than that for CTL [Emerson and Halpern 1985]. This, along with the fact that GCTL is at least exponentially more succinct than GμCALCULUS, makes GCTL even more appealing. The upper bound for the satisfiability complexity result is obtained by exploiting an automata-theoretic approach [Kupferman et al. 2000]. To develop a decision procedure for a logic with the tree model property, one first develops an appropriate notion of tree automata and studies their emptiness problem. Then, the satisfiability problem for the logic is reduced to the emptiness problem of the automata.

In Bianco et al. [2009], we first addressed the specific case of GCTL where numbers are coded in unary. In particular, it has first shown that unary GCTL indeed has the tree model property, by showing that any formula φ is satisfiable on a Kripke structure if and only if it has a tree model whose branching degree is polynomial in the size of φ. Then, a corresponding tree automaton model named *partitioning alternating Büchi tree automata* (PABT) has been introduced and shown that, for each unary GCTL formula φ, it is always possible to build in linear time a PABT accepting all tree models of φ. Then, by using a nontrivial extension of the Miyano and Hayashi technique [Miyano and Hayashi 1984] it has been shown an exponential translation of a PABT into a nondeterministic Büchi tree automata (NBT). Since the emptiness problem for NBT is solvable in polynomial time (in the size of the transition function that is polynomial in the number of states and exponential in the width of the tree in input) [Vardi and Wolper 1986], we obtain that the satisfiability problem for unary GCTL is solvable in EXPTIME.

A detailed analysis on this technique shows two points where it fails to give a single exponential-time algorithm when applied to binary GCTL. First, the tree model property shows for binary GCTL the necessity to consider also tree models with a branching degree exponential in the highest degree of the formula. Second, the number of states of the NBT derived from the PABT is double-exponential in the coding of the highest degree g of the formula. These two points reflect directly in the transition relation of the NBT, which turns to be double exponential in the coding of the degree g. To take care of the first point, we develop a sharp binary encoding of each tree model. In practice, for a given model $\mathcal{T}$ of φ we build a binary encoding $\mathcal{T}_D$ of $\mathcal{T}$, called *delayed generation tree*, such that, for each node x in $\mathcal{T}$ having $m+1$ children $x \cdot 0, \ldots, x \cdot m$, there is a corresponding node y of x in $\mathcal{T}_D$ and nodes $y \cdot 0^i$ having $x \cdot i$ as right child and $y \cdot 0^{(i+1)}$ as left child, for $0 \leq i \leq m$. To address the second point, we exploit a careful construction of the alternating automaton accepting all models of the formula, in a way that the graded numbers do not give any exponential blow-up in the translation of the automaton in an NBT.

We now describe the main idea behind the automata construction. Basically, we use alternating tree automata enriched with *satellites* (ATAS) as an extension of that introduced in Kupferman and Vardi [2006]. In particular, we use the Büchi acceptance condition (ABTS). The satellite is a nondeterministic tree automaton and is used to ensure that the tree model satisfies some structural properties along its paths and it is kept apart from the main automaton. This separation, as it has been proved in Kupferman and Vardi [2006], allows to solve the emptiness problem for Büchi automata in a time exponential in the number of states of the main automaton and polynomial in the number of states of the satellite. Then, we obtain the desired complexity by forcing the satellite to take care of the graded modalities and by noting that the main automaton is polynomial in the size of the formula.

The achieved result is even more appealing as we also show here that binary GCTL is much more succinct than GμCALCULUS. In particular, the best known translation from GCTL to GμCALCULUS is double-exponential in the degree of the formula [Bianco et al. 2010].

Related works. Graded modalities along with CTL have been also studied in Ferrante et al. [2008, 2009], but under a different semantics. There, the authors consider overlapping paths (as we do) as well as disjoint paths, but they neither consider the general framework of equivalence classes over paths nor the particular concepts of minimality and conservativeness, which we deeply analyze in our paper. In Ferrante et al. [2008] the model-checking problem for nonminimal and nonconservative unary GCTL has been investigated. In particular, by opportunely extending the classical algorithm for CTL [Clarke and Emerson 1981], they show that, in the case of overlapping paths, the model-checking problem is PTIME-COMPLETE (thus not harder than CTL), while in the case of disjoint paths, it is in PSPACE and both NPTIME-HARD and CONPTIME-HARD. The work continues in Ferrante et al. [2009], by showing a symbolic model-checking algorithm for the binary coding and, limited to the unary case, a satisfiability procedure. Regarding the comparison between GCTL and graded CTL with overlapping paths studied in Ferrante et al. [2008], it can be shown that they are equivalent by using an exponential reduction in both ways, whereas we do not know whether any of the two blow-ups can be avoided. However, it is important to note that our general technique can be also adapted to obtain an EXPTIME satisfiability procedure for the binary graded CTL under the semantics proposed in Ferrante et al. [2008]. Indeed, it is needed only to slightly modify the transition function of the main automaton (with respect to until and release formulas), without changing the structure of the whole satellite. Moreover, it can be used to prove that, in the case of unary GCTL, the complexity of the satisfiability problem is only polynomial in the degree. Finally, our method can be also applied to the satisfiability of the GμCALCULUS while the technique developed in Kupferman et al. [2002] cannot be used for GCTL.

Outline. In Section 2, we recall the basic notions regarding Kripke structures and trees, bisimulation, unwinding, and numeric partitions. Then we have Section 3, in which we introduce GCTL* and define its syntax and semantics, followed by Sections 4 and 5, in which there are studied the main properties of path equivalence relations and the particular case of the prefix path equivalence based on the concepts of minimality and conservativeness. In Section 6, we describe the ATAS automaton model. Finally, in Section 7 we construct the binary tree encoding of a Kripke structure and in Section 8 we describe the procedure used to solve the related satisfiability problem. Note that in the accompanying Appendix A we recall the classical mathematical notation and some basic definitions that are used throughout the whole article.

2. PRELIMINARIES

Kripke Structures. A *Kripke structure* (KS, for short) is a tuple $\mathcal{K} \triangleq \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, w_0\rangle$, where AP is a finite nonempty set of *atomic propositions*, W is an enumerable nonempty set of *worlds*, $w_0 \in \mathrm{W}$ is a designated *initial world*, $R \subseteq \mathrm{W} \times \mathrm{W}$ is a *transition relation*, and $\mathsf{L} : \mathrm{W} \to 2^{\mathrm{AP}}$ is a *labeling function* that maps each world to the set of atomic propositions true in that world. A KS is said to be *total* if and only if it has a total transition relation R, that is, for all $w \in \mathrm{W}$, there is $w' \in \mathrm{W}$ such that $(w, w') \in R$. By $\|\mathcal{K}\| \triangleq |R| \leq |\mathrm{W}|^2$ we denote the *size* of $\mathcal{K}$, which also corresponds to the size of the transition relation. A *finite* KS is a structure of finite size.

Kripke Trees. A *Kripke tree* (KT, for short) is a KS $\mathcal{T} \triangleq \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, \varepsilon\rangle$, where (i) $\mathrm{W} \subseteq \Delta^*$ is a Δ-tree for a given set Δ of directions and (ii), for all $t \in \mathrm{W}$ and $d \in \Delta$, it holds that $t \cdot d \in \mathrm{W}$ if and only if $(t, t \cdot d) \in R$.

Tracks and Paths. A *track* in $\mathcal{K}$ is a finite sequence of worlds $\rho \in \mathrm{W}^*$ such that, for all $i \in [0, |\rho| - 1[$, it holds that $((\rho)_i, (\rho)_{i+1}) \in R$. Furthermore, a *path* in $\mathcal{K}$ is a finite or infinite sequence of worlds $\pi \in \mathrm{W}^\infty$ such that, for all $i \in [0, |\pi| - 1[$, it holds that $((\pi)_i, (\pi)_{i+1}) \in R$ and if $|\pi| < \infty$ then there is no world $w \in \mathrm{W}$ such that $(\mathsf{lst}(\pi), w) \in R$, that is, it is *maximal*. Intuitively, tracks and paths of a KS $\mathcal{K}$ are legal sequences of reachable worlds in $\mathcal{K}$ that can be seen as a partial or complete description of the possible *computations* of the system modeled by $\mathcal{K}$. A track ρ is said to be *nontrivial* if and only if $|\rho| > 0$, that is, $\rho \neq \varepsilon$. We use $\mathrm{Trk}(\mathcal{K}) \subseteq \mathrm{W}^+$ and $\mathrm{Pth}(\mathcal{K}) \subseteq \mathrm{W}^\infty$ to indicate, respectively, the sets of all nontrivial tracks and paths of the KS $\mathcal{K}$. Moreover, by $\mathrm{Trk}(\mathcal{K}, w) \subseteq \mathrm{Trk}(\mathcal{K})$ and $\mathrm{Pth}(\mathcal{K}, w) \subseteq \mathrm{Pth}(\mathcal{K})$ we denote the subsets of tracks and paths starting at the world w.

Bisimulation. Let $\mathcal{K}_1 = \langle \mathrm{AP}, \mathrm{W}_1, R_1, \mathsf{L}_1, w_{0_1}\rangle$ and $\mathcal{K}_2 = \langle \mathrm{AP}, \mathrm{W}_2, R_2, \mathsf{L}_2, w_{0_2}\rangle$ be two KSs. Then, $\mathcal{K}_1$ and $\mathcal{K}_2$ are *bisimilar* if and only if there is a relation $\sim \subseteq \mathrm{W}_1 \times \mathrm{W}_2$ between worlds, called *bisimulation relation*, such that $w_{0_1} \sim w_{0_2}$ and if $w_1 \sim w_2$ then *(i)* $\mathsf{L}_1(w_1) = \mathsf{L}_2(w_2)$, *(ii)* for all $v_1 \in \mathrm{W}_1$ such that $(w_1, v_1) \in R_1$, there is $v_2 \in \mathrm{W}_2$ such that $(w_2, v_2) \in R_2$ and $v_1 \sim v_2$, and *(iii)* for all $v_2 \in \mathrm{W}_2$ such that $(w_2, v_2) \in R_2$, there is $v_1 \in \mathrm{W}_1$ such that $(w_1, v_1) \in R_1$ and $v_1 \sim v_2$.

Unwinding. Let $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, w_0\rangle$ be a KS. Then, the *unwinding* of $\mathcal{K}$ is the KT $\mathcal{K}_U \triangleq \langle \mathrm{AP}, \mathrm{W}', R', \mathsf{L}', \varepsilon\rangle$, where (i) W is the set of directions, (ii) the states in $\mathrm{W}' \triangleq \{\rho \in \mathrm{W}^* : w_0 \cdot \rho \in \mathrm{Trk}(\mathcal{K})\}$ are the suffixes of the tracks starting in w_0, (iii) $(\rho, \rho \cdot w) \in R'$ if and only if $(\mathsf{lst}(w_0 \cdot \rho), w) \in R$, and (iv) there is a surjective function $\mathsf{unw} : \mathrm{W}' \to \mathrm{W}$, called *unwinding function*, such that (iv.i) $\mathsf{unw}(\rho) \triangleq \mathsf{lst}(w_0 \cdot \rho)$ and (iv.ii) $\mathsf{L}'(\rho) \triangleq \mathsf{L}(\mathsf{unw}(\rho))$, for all $\rho \in \mathrm{W}'$ and $w \in \mathrm{W}$. It is easy to note that a KS is always bisimilar to its unwinding, since the unwinding function is a particular relation of bisimulation.

Numeric Partitions. Let $n \in [1, \omega[$. We define $\mathrm{P}(n)$ as the set of all *partition solutions* $p \in \mathbb{N}^n$ of the *linear Diophantine equation* $1 \cdot (p)_1 + 2 \cdot (p)_2 + \cdots + n \cdot (p)_n = n$ and $\mathrm{C}(n)$ as the set of all the *cumulative solutions* $c \in \mathbb{N}^{n+1}$ obtained by summing increasing sets of elements from p. Formally, $\mathrm{P}(n) \triangleq \{p \in \mathbb{N}^n : \sum_{i=1}^{n} i \cdot (p)_i = n\}$ and $\mathrm{C}(n) \triangleq \{c \in \mathbb{N}^{n+1} : \exists p \in \mathrm{P}(n).\ \forall i \in [1, n+1].\ (c)_i = \sum_{j=i}^{n} (p)_j\}$. It is easy to verify that all cumulative solutions satisfy the simple equation $(c)_1 + (c)_2 + \ldots + (c)_n = n$. Moreover, $(c)_i \geq (c)_{i+1}$, for all $i \in [1, n]$, and $(c)_{n+1} = 0$. Hence, there is just one cumulative solution $c \in \mathrm{C}(n)$, with $(c)_i = 1$, for all $i \in [1, n]$, which also corresponds to the unique solution $p \in \mathrm{P}(n)$, with $(p)_n = 1$. We use to define the cumulative solutions to be tuples of $n + 1$ and not only of n elements only to simplify the notation when we use this concept. As an example of these sets, consider the case $n = 4$. Then, we have that $\mathrm{P}(n) = \{(4, 0, 0, 0), (2, 1, 0, 0), (0, 2, 0, 0), (1, 0, 1, 0), (0, 0, 0, 1)\}$ and $\mathrm{C}(n) = \{(4, 0, 0, 0, 0), (3, 1, 0, 0, 0), (2, 2, 0, 0, 0), (2, 1, 1, 0, 0), (1, 1, 1, 1, 0)\}$. Note that $|\mathrm{C}(n)| = |\mathrm{P}(n)|$ and, since for each solution p of this Diophantine equation there is exactly one *partition* of n, we have that $|\mathrm{C}(n)| = \mathsf{p}(n)$, where $\mathsf{p}(n)$ is function returning the number of partitions of n. By [Apostol 1976] (see also Sloane and Plouffe [1995]), it holds that $\mathsf{p}(n) \to \frac{k_1}{n} \cdot 2^{k_2 \cdot \sqrt{n}}$, where $k_1 = 4 \cdot \sqrt{3}$ and $k_2 = \sqrt{2/3} \cdot \pi \cdot \log e$, for $n \to \infty$. Hence, $|\mathrm{C}(n)| = \Theta(\frac{1}{n} \cdot 2^{k_2 \cdot \sqrt{n}})$.

3. GRADED COMPUTATION TREE LOGICS

In this section, we introduce a class of extensions of the classical branching-time temporal logics CTL [Clarke and Emerson 1981] with graded path quantifiers. We show, in the next sections, that these extensions allow to gain expressiveness without paying any extra cost on deciding their satisfiability. To formally define the extended logics, we use the CTL* [Emerson and Halpern 1986] state and path formulas framework.

3.1. Syntax

The *graded full computation tree logic* (GCTL*, for short) extends CTL* by using two special path quantifiers, the existential $\mathsf{E}^{\geq g}$ and the universal $\mathsf{A}^{<g}$, where the finite or infinite number $g \in \widehat{\mathbb{N}}$ denotes the corresponding *degree*. As in CTL*, these quantifiers can prefix a linear-time formula composed of an arbitrary Boolean combination and nesting of temporal operators X *"next"*, U *"until"*, and R *"release"* together with their weak version $\tilde{\mathsf{X}}$, $\tilde{\mathsf{U}}$, and $\tilde{\mathsf{R}}$. The quantifiers $\mathsf{E}^{\geq g}$ and $\mathsf{A}^{<g}$ can be informally read as "there are at least g paths" and "all but less than g paths", respectively. The formal syntax of GCTL* follows.

Definition 3.1 (GCTL* *Syntax*). GCTL* *state* (φ) and *path* (ψ) *formulas* are built inductively from the sets of atomic propositions AP in the following way, where $p \in \mathrm{AP}$ and $g \in \widehat{\mathbb{N}}$:

(1) $\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{E}^{\geq g}\psi \mid \mathsf{A}^{<g}\psi$;
(2) $\psi ::= \varphi \mid \neg\psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \mathsf{X}\,\psi \mid \psi\,\mathsf{U}\,\psi \mid \psi\,\mathsf{R}\,\psi \mid \tilde{\mathsf{X}}\,\psi \mid \psi\,\tilde{\mathsf{U}}\,\psi \mid \psi\,\tilde{\mathsf{R}}\,\psi$.

The class of GCTL* formulas is the set of state formulas generated by the given grammar. In addition, the simpler class of GCTL formulas is obtained by forcing each temporal operator occurring into a formula to be coupled with a path quantifier, as in the classical case of CTL.

We now introduce some auxiliary syntactical notation. For a formula φ, we define the *degree* $\mathring{\varphi}$ of φ as the maximum natural number g occurring among the degrees of all its path quantifiers. Formally, (i) $\mathring{p} \triangleq 0$, for $p \in \mathrm{AP}$, (ii) $(\mathring{\mathsf{Op}}\,\psi) \triangleq \mathring{\psi}$, for all $\mathsf{Op} \in \{\neg, \mathsf{X}, \tilde{\mathsf{X}}\}$, (iii) $(\psi_1 \mathring{\mathsf{Op}}\,\psi_2) \triangleq \max\{\mathring{\psi}_1, \mathring{\psi}_2\}$, for all $\mathsf{Op} \in \{\wedge, \vee, \mathsf{U}, \mathsf{R}, \tilde{\mathsf{U}}, \tilde{\mathsf{R}}\}$, *(iv)* $(\mathring{\mathsf{Qn}}\,\psi) \triangleq \max\{g, \mathring{\psi}\}$, for all $\mathsf{Qn} \in \{\mathsf{E}^{\geq g}, \mathsf{A}^{<g}\}$ with $g \in \mathbb{N}$, and *(v)* $(\mathring{\mathsf{Qn}}\ \psi) \triangleq \mathring{\psi}$, for all $\mathsf{Qn} \in \{\mathsf{E}^{\geq \omega}, \mathsf{A}^{<\omega}\}$. We assume that the degree is coded in binary. The *length* of φ, denoted by $|\varphi|$, is defined as for CTL* and does not consider the degrees at all. Formally, (i) $|p| \triangleq 1$, for $p \in \mathrm{AP}$, (ii) $|\mathsf{Op}\ \psi| \triangleq 1+|\psi|$, for all $\mathsf{Op} \in \{\neg, \mathsf{X}, \tilde{\mathsf{X}}\}$, (iii) $|\psi_1 \mathsf{Op}\, \psi_2| \triangleq 1+|\psi_1|+|\psi_2|$, for all $\mathsf{Op} \in \{\wedge, \vee, \mathsf{U}, \mathsf{R}, \tilde{\mathsf{U}}, \tilde{\mathsf{R}}\}$, and *(iv)* $|\mathsf{Qn}\ \psi| \triangleq 1 + |\psi|$, for all $\mathsf{Qn} \in \{\mathsf{E}^{\geq g}, \mathsf{A}^{<g}\}$. Accordingly, the *size* of φ, denoted by $\|\varphi\|$, is defined in the same way of the length, by considering $\|\mathsf{E}^{\geq g}\psi\|$ and $\|\mathsf{A}^{<g}\psi\|$ to be equal to $1 + \lceil \log(g) \rceil + \|\psi\|$, for $g \in [1, \omega[$, and to $1 + \|\psi\|$, otherwise. Clearly, it holds that $\lceil \log(\mathring{\varphi}) \rceil \leq \|\varphi\|$ and $|\varphi| \leq \|\varphi\|$. We also use $\mathsf{cl}(\psi)$ to denote the classical Fischer-Ladner *closure* [Fischer and Ladner 1979] of ψ defined recursively as for CTL* in the following way: $\mathsf{cl}(\varphi) \triangleq \{\varphi\} \cup \mathsf{cl}'(\varphi)$, for all state formulas φ and $\mathsf{cl}(\psi) \triangleq \mathsf{cl}'(\psi)$, for all path formulas ψ, where (i) $\mathsf{cl}'(p) \triangleq \emptyset$, for $p \in \mathrm{AP}$, (ii) $\mathsf{cl}'(\mathsf{Op}\ \psi) \triangleq \mathsf{cl}(\psi)$, for all $\mathsf{Op} \in \{\neg, \mathsf{X}, \tilde{\mathsf{X}}\}$, (iii) $\mathsf{cl}'(\psi_1 \mathsf{Op}\, \psi_2) \triangleq \mathsf{cl}(\psi_1) \cup \mathsf{cl}(\psi_2)$, for all $\mathsf{Op} \in \{\wedge, \vee, \mathsf{U}, \mathsf{R}, \tilde{\mathsf{U}}, \tilde{\mathsf{R}}\}$, and (iv) $\mathsf{cl}'(\mathsf{Qn}\ \psi) \triangleq \mathsf{cl}(\psi)$, for all $\mathsf{Qn} \in \{\mathsf{E}^{\geq g}, \mathsf{A}^{<g}\}$. Intuitively, $\mathsf{cl}(\varphi)$ is the set of all the state formulas that are subformulas of φ. Finally, by $\mathsf{rcl}(\psi)$ we denote the *reduced closure* of ψ, that is, the set of the maximal states formulas contained in ψ. Formally, (i) $\mathsf{rcl}(\varphi) \triangleq \{\varphi\}$, for all state formulas φ, (ii) $\mathsf{rcl}(\mathsf{Op}\ \psi) \triangleq \mathsf{rcl}(\psi)$ when $\mathsf{Op}\ \psi$ is a path formula, for all $\mathsf{Op} \in \{\neg, \mathsf{X}, \tilde{\mathsf{X}}\}$, and (iii)8 $\mathsf{rcl}(\psi_1 \mathsf{Op}\ \psi_2) \triangleq \mathsf{rcl}(\psi_1) \cup \mathsf{rcl}(\psi_2)$ when $\psi_1 \mathsf{Op}\ \psi_2$ is a path formula, for all $\mathsf{Op} \in \{\wedge, \vee, \mathsf{U}, \mathsf{R}, \tilde{\mathsf{U}}, \tilde{\mathsf{R}}\}$. It is immediate to see that $\mathsf{rcl}(\psi) \subseteq \mathsf{cl}(\psi)$ and $|\mathsf{cl}(\psi)| = \mathrm{O}(|\psi|)$.

3.2. Semantics

We now define the semantics of GCTL* w.r.t. a Ks $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, w_0 \rangle$. For a world $w \in \mathrm{W}$, we write $\mathcal{K}, w \models \varphi$ to indicate that a state formula φ holds on $\mathcal{K}$ at w. Moreover, for a path $\pi \in \mathrm{Pth}(\mathcal{K})$, we write $\mathcal{K}, \pi \models \psi$ to indicate that a path formula ψ holds on π. The semantics of GCTL* state formulas simply extends that of CTL* and is reported in the following. In particular, for the definition of graded quantifiers, we deeply make use of a generic equivalence relation $\equiv_{\mathcal{K}}^{\psi}$ on the set of paths $\mathrm{Pth}(\mathcal{K})$ that may depend on both the Ks $\mathcal{K}$ and the path formula ψ. This equivalence is used to reasonably count the number of ways a structure has to satisfy a path formula starting from a given node, w.r.t. an a priori fixed criterion. The semantics of the GCTL* path formulas is defined as usual for LTL on both finite and infinite paths and, for sake of simplicity, is omitted here. We recall that the weak temporal operators are used to deal with finite paths on which their strong version may result unsatisfiable (see Eisner et al. [2003] and Appendix B for a full definition of the LTL semantics with strong and weak temporal operators).

Definition 3.2 (GCTL* *Semantics*). Given a Ks $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, w_0 \rangle$, for all GCTL* state formulas φ and worlds $w \in \mathrm{W}$, the relation $\mathcal{K}, w \models \varphi$ is inductively defined as follows.

(1) $\mathcal{K}, w \models p$ if and only if $p \in \mathsf{L}(w)$, with $p \in \mathrm{AP}$.
(2) For all state formulas φ, φ_1, and φ_2, it holds that:
  (a) $\mathcal{K}, w \models \neg\varphi$ if and only if not $\mathcal{K}, w \models \varphi$, that is $\mathcal{K}, w \not\models \varphi$;
  (b) $\mathcal{K}, w \models \varphi_1 \wedge \varphi_2$ if and only if $\mathcal{K}, w \models \varphi_1$ and $\mathcal{K}, w \models \varphi_2$;
  (c) $\mathcal{K}, w \models \varphi_1 \vee \varphi_2$ if and only if $\mathcal{K}, w \models \varphi_1$ or $\mathcal{K}, w \models \varphi_2$.
(3) For a number $g \in \widehat{\mathbb{N}}$ and a path formula ψ, it holds that:
  (a) $\mathcal{K}, w \models \mathsf{E}^{\geq g}\psi$ if and only if $|(\mathrm{Pth}(\mathcal{K}, w, \psi)/\equiv_{\mathcal{K}}^{\psi})| \geq g$;
  (b) $\mathcal{K}, w \models \mathsf{A}^{<g}\psi$ if and only if $|(\mathrm{Pth}(\mathcal{K}, w, \neg\psi)/\equiv_{\mathcal{K}}^{\neg\psi})| < g$;
  where $\mathrm{Pth}(\mathcal{K}, w, \psi) \triangleq \{\pi \in \mathrm{Pth}(\mathcal{K}, w) : \mathcal{K}, \pi \models \psi\}$ is the set of paths of $\mathcal{K}$ starting in w that satisfy the path formula ψ and $(\mathrm{Pth}(\mathcal{K}, w, \psi)/\equiv_{\mathcal{K}}^{\psi})$ denotes the *quotient* set of $\mathrm{Pth}(\mathcal{K}, w, \psi)$ w.r.t. the equivalence relation $\equiv_{\mathcal{K}}^{\psi}$, that is, the set of all the related equivalence classes.

For all GCTL* path formulas ψ and paths $\pi \in \mathrm{Pth}(\mathcal{K})$, the relation $\mathcal{K}, \pi \models \psi$ is defined as follows.

(4) $\mathcal{K}, \pi \models \psi$ if and only if $\varpi_{\mathcal{K},\psi}(\pi) \models \psi$, where ψ is considered as an LTL formula over its restricted closure $\mathsf{rcl}(\psi)$ and $\varpi_{\mathcal{K},\psi}(\pi) \in (2^{\mathsf{rcl}(\psi)})^{|\pi|}$ is the trace such that $\varphi \in (\varpi_{\mathcal{K},\psi}(\pi))_k$ if and only if $\mathcal{K}, (\pi)_k \models \varphi$, for all $\varphi \in \mathsf{rcl}(\psi)$ and $k \in [0, |\pi|[$.

Intuitively, by using the graded existential quantifier $\mathsf{E}^{\geq g}\psi$, we can count how many different equivalence classes w.r.t. $\equiv_{\mathcal{K}}^{\psi}$ there are over the set $\mathrm{Pth}(\mathcal{K}, w, \psi)$ of paths satisfying ψ. The universal quantifier $\mathsf{A}^{<g}\psi$ is simply the dual of $\mathsf{E}^{\geq g}\psi$ and it allows to count how many classes w.r.t. $\equiv_{\mathcal{K}}^{\neg\psi}$ there are over the set $\mathrm{Pth}(\mathcal{K}, w, \neg\psi)$ of paths not satisfying ψ. It is important to note that, since $(\mathrm{Pth}(\mathcal{K}, w, \psi)/\equiv_{\mathcal{K}}^{\psi}) \neq \emptyset$ and $(\mathrm{Pth}(\mathcal{K}, w, \neg\psi) /\equiv_{\mathcal{K}}^{\neg\psi}) \neq \emptyset)$ are equivalent to $\mathrm{Pth}(\mathcal{K}, w, \psi) \neq \emptyset$ and $\mathrm{Pth}(\mathcal{K}, w, \neg\psi) \neq \emptyset$, respectively, it holds that all GCTL* formulas with degree 1 are CTL* formulas too, and vice versa.

Observe that, in the definition of the semantics, we introduced a transformation $\varpi_{\mathcal{K},\psi}(\cdot)$, for each path formula ψ, that maps each path π of the Ks $\mathcal{K}$ to a trace $\varpi_{\mathcal{K},\psi}(\pi) \in (2^{\mathsf{rcl}(\psi)})^{|\pi|}$ given by the sequence of sets of state formulas in $\mathsf{rcl}(\psi)$ satisfied at the worlds of π. Hence, we interpret the path formula ψ on AP evaluated on π as an LTL formula on $\mathsf{rcl}(\psi)$ evaluated on $\varpi_{\mathcal{K},\psi}(\pi)$.

Let $\mathcal{K}$ be a Ks and φ be a GCTL* formula. Then, $\mathcal{K}$ is a *model* for φ, in symbols $\mathcal{K} \models \varphi$, if and only if $\mathcal{K}, w_0 \models \varphi$, where we recall that w_0 is the initial state of $\mathcal{K}$. In this case, we also say that $\mathcal{K}$ is a model for φ on w_0. A formula φ is said to be *satisfiable* if and only if there exists a model for it. Moreover, it is an *invariant* for the two Kss $\mathcal{K}_1$ and $\mathcal{K}_2$ if and only if either $\mathcal{K}_1 \models \varphi$ and $\mathcal{K}_2 \models \varphi$ or $\mathcal{K}_1 \not\models \varphi$ and $\mathcal{K}_2 \not\models \varphi$. For all state formulas φ_1 and φ_2, we say that φ_1 *implies* φ_2, in symbols $\varphi_1 \Rightarrow \varphi_2$, if and only if, for all Ks $\mathcal{K}$, it holds that if $\mathcal{K} \models \varphi_1$ then $\mathcal{K} \models \varphi_2$. Consequently, we say that φ_1 is *equivalent* to φ_2, in symbols $\varphi_1 \equiv \varphi_2$, if and only if $\varphi_1 \Rightarrow \varphi_2$ and $\varphi_2 \Rightarrow \varphi_1$. In the following, when we say that two GCTL* paths formulas ψ_1 and ψ_2 are equivalent, in symbols $\psi_1 \equiv \psi_2$, we mean that they are equivalent if considered as LTL formulas over the union $\mathsf{rcl}(\psi_1) \cup \mathsf{rcl}(\psi_2)$ of their restricted closures.

For technical reasons, we also define the relation of satisfiability of path formulas on tracks, by simply setting $\mathcal{K}, \rho \models \psi$ if and only if $\varpi_{\mathcal{K},\psi}(\rho) \models \psi$, for all $\rho \in \mathrm{Trk}(\mathcal{K})$. We now show the basic properties of the satisfiability relation $\models$ on paths and tracks directly inherited by the LTL semantics.

PROPOSITION 3.1 (PATH SATISFIABILITY PROPERTIES). *Let φ be a state formula, ψ, ψ_1, and ψ_2 be path formulas, and $\pi \in (\mathrm{Pth}(\mathcal{K}, w) \cup \mathrm{Trk}(\mathcal{K}, w))$ be a path/track starting at the world w of the* Ks $\mathcal{K}$. *Then, the following properties hold:* (i) *if $\psi_1 \equiv \psi_2$ then $\mathcal{K}, \pi \models \psi_1$ if and only if $\mathcal{K}, \pi \models \psi_2$;* (ii) *$\mathcal{K}, w \models \varphi$ if and only if $\mathcal{K}, \pi \models \varphi$;* (iii) *$\mathcal{K}, \pi \models \psi_1 \wedge \psi_2$ if and only if $\mathcal{K}, \pi \models \psi_1$ and $\mathcal{K}, \pi \models \psi_2$;* (iv) *$\mathcal{K}, \pi \models \psi_1 \vee \psi_2$ if and only if $\mathcal{K}, \pi \models \psi_1$ or $\mathcal{K}, \pi \models \psi_2$;* (v) *$\mathcal{K}, \pi \models \mathsf{X}\,\psi$ if and only if $\pi_{\geq 1} \neq \varepsilon$ and $\mathcal{K}, \pi_{\geq 1} \models \psi$;* (vi) *$\mathcal{K}, \pi \models \tilde{\mathsf{X}}\,\psi$ if and only if $\pi_{\geq 1} = \varepsilon$ or $\mathcal{K}, \pi_{\geq 1} \models \psi$;* (vii) *$\mathcal{K}, \pi \models \psi_1 \mathsf{U}\, \psi_2$ if and only if $\mathcal{K}, \pi \models \psi_2 \vee \psi_1 \wedge \mathsf{X}\, \psi_1 \mathsf{U}\, \psi_2$;* (viii) *$\mathcal{K}, \pi \models \psi_1 \mathsf{R}\, \psi_2$ if and only if $\mathcal{K}, \pi \models \psi_2 \wedge (\psi_1 \vee \mathsf{X}\, \psi_1 \mathsf{R}\, \psi_2)$;* (ix) *$\mathcal{K}, \pi \models \psi_1 \tilde{\mathsf{U}}\, \psi_2$ if and only if $\mathcal{K}, \pi \models \psi_2 \vee \psi_1 \wedge \tilde{\mathsf{X}}\, \psi_1 \tilde{\mathsf{U}}\, \psi_2$;* (x) *$\mathcal{K}, \pi \models \psi_1 \tilde{\mathsf{R}}\, \psi_2$ if and only if $\mathcal{K}, \pi \models \psi_2 \wedge (\psi_1 \vee \tilde{\mathsf{X}}\, \psi_1 \tilde{\mathsf{R}}\, \psi_2)$.*

PROOF. First note that in this proof, we make use of a slightly more general map of $\varpi_{\mathcal{K},\psi}(\cdot)$ that associates each path in $\mathcal{K}$ with the sequence of state formulas, belonging to a given set Z, satisfied at the worlds of π. Formally, by $\varpi_{\mathcal{K},\mathrm{Z}}(\pi)$ we denote the trace in $(2^{\mathrm{Z}})^{|\pi|}$ such that, for all $\varphi \in \mathrm{Z}$ and $k \in [0, |\pi|[$, it holds that $\varphi \in (\varpi_{\mathcal{K},\mathrm{Z}}(\pi))_k$ if and only if $\mathcal{K}, (\pi)_k \models \varphi$. Observe that, for every GCTL* path formula ψ and set Z of state formulas containing $\mathsf{rcl}(\psi)$, when ψ is interpreted as an LTL formula on $\mathsf{rcl}(\psi)$, it is satisfied on a trace $\varpi_{\mathcal{K},\psi}(\pi)$ if and only if it is satisfied on all traces $\varpi_{\mathcal{K},\mathrm{Z}}(\pi)$ as well. We can now start with the proofs of all items.

(i) Let $Z = \mathsf{rcl}(\psi_1) \cup \mathsf{rcl}(\psi_2)$. For $i \in \{1, 2\}$, if $\mathcal{K}, \pi \models \psi_i$, then $\varpi_{\mathcal{K},\psi_i}(\pi) \models \psi_i$. Now, since $\mathsf{rcl}(\psi_i) \subseteq Z$, we have that $\varpi_{\mathcal{K},Z}(\pi) \models \psi_i$. By the equivalence $\psi_1 \equiv \psi_2$, we obtain then that $\varpi_{\mathcal{K},Z}(\pi) \models \psi_{3-i}$. So, since $\mathsf{rcl}(\psi_{3-i}) \subseteq Z$, we have that $\varpi_{\mathcal{K},\psi_{3-i}}(\pi) \models \psi_{3-i}$ and consequently $\mathcal{K}, \pi \models \psi_{3-i}$.

(ii) Since φ is a state formula, by definition of the transformation map $\varpi_{\mathcal{K},\varphi}(\cdot)$, we have that $\mathcal{K}, w \models \varphi$ if and only if $\varphi \in (\varpi_{\mathcal{K},\varphi}(\pi))_0$ and so $\varpi_{\mathcal{K},\varphi}(\pi) \models \varphi$, from which we derive $\mathcal{K}, \pi \models \varphi$ and vice versa.

(iii) Let $\psi = \psi_1 \wedge \psi_2$. Then, it holds that $\mathcal{K}, \pi \models \psi$ if and only if $\varpi_{\mathcal{K},\psi}(\pi) \models \psi$, which is equivalent to $\varpi_{\mathcal{K},\psi}(\pi) \models \psi_i$, for $i \in \{1, 2\}$. At this point, since $\mathsf{rcl}(\psi_i) \subseteq \mathsf{rcl}(\psi)$, we have that $\mathcal{K}, \pi \models \psi$ is equivalent to $\varpi_{\mathcal{K},\psi_i}(\pi) \models \psi_i$, for $i \in \{1, 2\}$. Hence, $\mathcal{K}, \pi \models \psi$ if and only if $\mathcal{K}, \pi \models \psi_1$ and $\mathcal{K}, \pi \models \psi_2$.

(iv) Mutatis mutandis, the proof is the same of the previous item.

(v) Note that $\mathsf{rcl}(\mathsf{X}\,\psi) = \mathsf{rcl}(\psi)$. Then, it holds that $\mathcal{K}, \pi \models \mathsf{X}\,\psi$ if and only if $\varpi_{\mathcal{K},\psi}(\pi) \models \mathsf{X}\,\psi$, which is equivalent to $(\varpi_{\mathcal{K},\psi}(\pi))_{\geq 1} \neq \varepsilon$, that is, $\pi_{\geq 1} \neq \varepsilon$, and $(\varpi_{\mathcal{K},\psi}(\pi))_{\geq 1} \models \psi$, that is, $\varpi_{\mathcal{K},\psi}(\pi_{\geq 1}) \models \psi$. Hence, $\mathcal{K}, \pi \models \mathsf{X}\,\psi$ if and only if $\pi_{\geq 1} \neq \varepsilon$ and $\mathcal{K}, \pi_{\geq 1} \models \psi$.

(vi) Mutatis mutandis, the proof is the same of the previous item.

(vii) These items can be directly derived by Item i and the classical LTL one step unfolding equivalences $\psi_1 \mathsf{U}\, \psi_2 \equiv \psi_2 \vee \psi_1 \wedge \mathsf{X}\, \psi_1 \mathsf{U}\, \psi_2$, $\psi_1 \mathsf{R}\, \psi_2 \equiv \psi_2 \wedge (\psi_1 \vee \mathsf{X}\, \psi_1 \mathsf{R}\, \psi_2)$, $\psi_1 \tilde{\mathsf{U}}\, \psi_2 \equiv \psi_2 \vee \psi_1 \wedge \tilde{\mathsf{X}}\, \psi_1 \tilde{\mathsf{U}}\, \psi_2$, and $\psi_1 \tilde{\mathsf{R}}\, \psi_2 \equiv \psi_2 \wedge (\psi_1 \vee \tilde{\mathsf{X}}\, \psi_1 \tilde{\mathsf{R}}\, \psi_2)$. □

In the rest of the article, we consider only formulas in *positive normal form* (*pnf*, for short), that is, the negation is applied only to atomic propositions. In fact, it is to this aim that we have considered in the syntax of GCTL* both the Boolean connectives $\wedge$ and $\vee$, the path quantifiers $\mathsf{A}^{<g}$ and $\mathsf{E}^{\geq g}$, and temporal operators X, U, and R together with their weak version $\tilde{\mathsf{X}}$, $\tilde{\mathsf{U}}$, and $\tilde{\mathsf{R}}$. Indeed, all formulas can be linearly translated in *pnf* by using De Morgan's laws and the following equivalences, which directly follow from the semantics of the logic: $\neg \mathsf{E}^{\geq g}\psi \equiv \mathsf{A}^{<g}\neg\psi$; $\neg \mathsf{X}\,\psi \equiv \tilde{\mathsf{X}}\,\neg\psi$; $\neg(\psi_1 \mathsf{U}\,\psi_2) \equiv (\neg\psi_1)\tilde{\mathsf{R}}(\neg\psi_2)$; $\neg(\psi_1 \mathsf{R}\,\psi_2) \equiv (\neg\psi_1)\tilde{\mathsf{U}}(\neg\psi_2)$. Under this assumption, we consider $\neg\varphi$ as the *pnf* formula equivalent to the negation of φ. Finally, as abbreviations we use the Boolean values $\mathfrak{t}$ ("true") and $\mathfrak{f}$ ("false") and the path quantifiers $\mathsf{E}^{>g}\psi \triangleq \mathsf{E}^{\geq g+1}\psi$ ("there exist more than g paths"), $\mathsf{A}^{\leq g}\psi \triangleq \mathsf{A}^{<g+1}\psi$ ("all but at most g paths"), $\mathsf{E}^{=g}\psi \triangleq \mathsf{E}^{\geq g}\psi \wedge \neg\mathsf{E}^{>g}\psi$ ("there exist just g paths"), and $\mathsf{A}^{=g}\psi \triangleq \mathsf{A}^{\leq g}\psi \wedge \neg\mathsf{A}^{<g}\psi$ ("all but exactly g paths"), with $g \in [0, \omega[$.

We now report some basic equivalences that are directly derived from the definition of the logic and Proposition 3.1, and are independent from the particular path equivalence relation $\equiv_{\cdot}$ considered.

PROPOSITION 3.2 (BASIC EQUIVALENCES). *Let φ and ψ be a state and a path formula, respectively, and $g \in \widehat{\mathbb{N}}$. Then, the following equivalences hold:* (i) $\mathsf{E}^{\geq 0}\psi \equiv \mathfrak{t}$; (ii) $\mathsf{E}^{\geq 1}\varphi \equiv \varphi$; (iii) $\mathsf{E}^{\geq 1}(\varphi \wedge \psi) \equiv \varphi \wedge \mathsf{E}^{\geq 1}\psi$; (iv) $\mathsf{E}^{\geq 1}(\varphi \vee \psi) \equiv \varphi \vee \mathsf{E}^{\geq 1}\psi$; (v) $\mathsf{E}^{\geq 1}\mathsf{X}\,\psi \equiv \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq 1}\psi$; (vi) $\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\psi \equiv \mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}^{\geq 1}\mathsf{X}\,\psi$; (vii) $\mathsf{E}^{>g}\psi \Rightarrow \mathsf{E}^{\geq g}\psi$; (viii) $\mathsf{A}^{<0}\psi \equiv \mathfrak{f}$; (ix) $\mathsf{A}^{<1}\varphi \equiv \varphi$; (x) $\mathsf{A}^{<1}(\varphi \wedge \psi) \equiv \varphi \wedge \mathsf{A}^{<1}\psi$; (xi) $\mathsf{A}^{<1}(\varphi \vee \psi) \equiv \varphi \vee \mathsf{A}^{<1}\psi$; (xii) $\mathsf{A}^{<1}\mathsf{X}\,\psi \equiv \mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t} \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\,\psi$; (xiii) $\mathsf{A}^{<1}\tilde{\mathsf{X}}\,\psi \equiv \mathsf{A}^{<1}\tilde{\mathsf{X}}\,\mathsf{A}^{<1}\psi$; (xiv) $\mathsf{A}^{<g}\psi \Rightarrow \mathsf{A}^{\leq g}\psi$.

Finally, we list the classical CTL fixpoint equivalences embedded in the GCTL framework, for the four binary temporal operators U, R, $\tilde{\mathsf{U}}$, and $\tilde{\mathsf{R}}$.

PROPOSITION 3.3 (CTL FIXPOINT EQUIVALENCES). *Let φ_1 and φ_2 be two state formulas. Then, the following hold:*

(i) $\mathsf{E}^{\geq 1}\varphi_1 \mathsf{U}\,\varphi_2 \equiv \varphi_2 \vee \varphi_1 \wedge \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq 1}\varphi_1 \mathsf{U}\,\varphi_2$;
(ii) $\mathsf{E}^{\geq 1}\varphi_1 \mathsf{R}\,\varphi_2 \equiv \varphi_2 \wedge (\varphi_1 \vee \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq 1}\varphi_1 \mathsf{R}\,\varphi_2)$;
(iii) $\mathsf{E}^{\geq 1}\varphi_1 \tilde{\mathsf{U}}\,\varphi_2 \equiv \varphi_2 \vee \varphi_1 \wedge (\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq 1}\varphi_1 \tilde{\mathsf{U}}\,\varphi_2)$;
(iv) $\mathsf{E}^{\geq 1}\varphi_1 \tilde{\mathsf{R}}\,\varphi_2 \equiv \varphi_2 \wedge (\varphi_1 \vee \mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq 1}\varphi_1 \tilde{\mathsf{R}}\,\varphi_2)$;

(v) $\mathsf{A}^{<1}\varphi_1 \mathsf{U}\, \varphi_2 \equiv \varphi_2 \vee \varphi_1 \wedge (\mathsf{A}^{<1}\mathsf{X}\, \mathfrak{t} \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\, \mathsf{A}^{<1}\varphi_1 \mathsf{U}\, \varphi_2)$;
(vi) $\mathsf{A}^{<1}\varphi_1 \mathsf{R}\, \varphi_2 \equiv \varphi_2 \wedge (\varphi_1 \vee \mathsf{A}^{<1}\mathsf{X}\, \mathfrak{t} \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\, \mathsf{A}^{<1}\varphi_1 \mathsf{R}\, \varphi_2)$;
(vii) $\mathsf{A}^{<1}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2 \equiv \varphi_2 \vee \varphi_1 \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\, \mathsf{A}^{<1}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2$;
(viii) $\mathsf{A}^{<1}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2 \equiv \varphi_2 \wedge (\varphi_1 \vee \mathsf{A}^{<1}\tilde{\mathsf{X}}\, \mathsf{A}^{<1}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2)$.

4. PATH EQUIVALENCE PROPERTIES

In the definition of GCTL* semantics, we make use of an arbitrary equivalence relation on paths. It is useful to investigate what properties can make such an equivalence a reasonable one for our purposes. In this section, we present a detailed exposition of its principal properties. Note that, in order to be not too repetitive, when we talk about "number of paths", we always mean the number of equivalence classes of paths w.r.t. a path formula, which is clear from the context. Moreover, every equivalence concerning the universal quantifier, if not otherwise specified, is obtained through the dualization ($\mathsf{A}^{<g}\psi \equiv \neg \mathsf{E}^{\geq g}\neg\psi$) of the related existential one.

To help the reader in following the exposition of the several properties we are going to introduce, we now give an outline of this section. First, in Section 4.1, we introduce two basic properties of equivalences on paths, namely, *syntax independence* and *state focus*. In Section 4.2, we further give the properties of *next consistency*, along with its weak form, and that of *source dependence*. From this we derive two fundamental expansion constructions that allow to generalize, to the case of graded quantifiers, the classical CTL* expansion equivalences $\mathsf{EX}\,\psi \equiv \mathsf{EX}\,\mathsf{E}\psi$ and $\mathsf{A}\tilde{\mathsf{X}}\,\psi \equiv \mathsf{A}\tilde{\mathsf{X}}\,\mathsf{A}\psi$. In Section 4.3, we introduce two consistency properties on the Boolean connectives $\wedge$ and $\vee$ and a satisfiability constraint. Finally, in Section 4.4, we define the concept of *adequacy* of an equivalence on paths and use it to show a set of fixpoint equivalences for GCTL, which are used, in Section 5, to prove the existence of a translation of a fragment of GCTL into GμCalculus.

4.1. Elementary Requirements

Suppose we have two equivalent path formulas ψ_1 and ψ_2. Then, we would like to have them to be exchangeable in a GCTL* path quantification, obtaining in this way that two state formulas $\mathsf{Qn}\,\psi_1$ and $\mathsf{Qn}\,\psi_2$ are equivalent, for all $\mathsf{Qn} \in \{\mathsf{E}^{\geq n}, \mathsf{A}^{<n}\}$ and $n \in \widehat{\mathbb{N}}$. Hence, what we need to require is that, whenever two paths are equivalent w.r.t. ψ_1, they are equivalent w.r.t. ψ_2, too. Before introducing the formal definition of this concept, we want to enlighten on the equivalence relation between path formulas we consider here is not just the classical LTL equivalence $\equiv$, but rather a generic equivalence $\cong_{\mathcal{K}}^{w}$ that may depend on both a Ks and one of its worlds. More motivations for this choice are given in the next subsection.

Definition 4.1 (*Syntax Independence*). An equivalence relation $\equiv_{\mathcal{K}}^{\cdot}$ on paths is said to be *syntax independent* if and only if, for all pairs of equivalent path formulas ψ_1 and ψ_2 w.r.t $\cong_{\mathcal{K}}^{w}$, it holds that $\pi_1 \equiv_{\mathcal{K}}^{\psi_1} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi_2} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w)$.

THEOREM 4.1 (EQUIVALENT QUANTIFICATIONS). *Let $\equiv_{\cdot}^{\cdot}$ be a syntax-independent equivalence relation. Moreover, let ψ_1 and ψ_2 be two equivalent path formulas and $g \in \widehat{\mathbb{N}}$. Then, the following holds:* (i) $\mathsf{E}^{\geq g}\psi_1 \equiv \mathsf{E}^{\geq g}\psi_2$ *and* (ii) $\mathsf{A}^{<g}\psi_1 \equiv \mathsf{A}^{<g}\psi_2$.

PROOF. Let $\mathcal{K}$ be a Ks and w_0 its initial world. Since $\psi_1 \equiv \psi_2$, by Item i of Proposition 3.1, it is immediate to see that $\mathrm{Pth}(\mathcal{K}, w_0, \psi_1) = \mathrm{Pth}(\mathcal{K}, w_0, \psi_2)$ and consequently $(\mathrm{Pth}(\mathcal{K}, w_0, \psi_1)/{\equiv_{\mathcal{K}}^{\psi_1}}) = (\mathrm{Pth}(\mathcal{K}, w_0, \psi_2)/{\equiv_{\mathcal{K}}^{\psi_1}})$. Now, by the syntax-independence property, we have that $\pi_1 \equiv_{\mathcal{K}}^{\psi_1} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi_2} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K})$. Thus, we have that $(\mathrm{Pth}(\mathcal{K}, w_0, \psi_2)/{\equiv_{\mathcal{K}}^{\psi_1}}) = (\mathrm{Pth}(\mathcal{K}, w_0, \psi_2)/{\equiv_{\mathcal{K}}^{\psi_2}})$. Hence the thesis. $\square$

The following corollary is directly derived by using the classical LTL equivalences for the four binary temporal operators.

COROLLARY 4.1 (ONE STEP UNFOLDING). *Let* $\equiv_{\cdot}$ *be a syntax-independent equivalence relation. Moreover, let* ψ_1 *and* ψ_2 *be two path formulas and* $g \in \widehat{\mathbb{N}}$*. Then, the following equivalences hold:* (i) $\mathsf{E}^{\geq g}\psi_1 \mathsf{U}\, \psi_2 \equiv \mathsf{E}^{\geq g}(\psi_2 \vee \psi_1 \wedge \mathsf{X}\, \psi_1 \mathsf{U}\, \psi_2)$*;* (ii) $\mathsf{E}^{\geq g}\psi_1 \mathsf{R}\, \psi_2 \equiv \mathsf{E}^{\geq g}(\psi_2 \wedge (\psi_1 \vee \mathsf{X}\, \psi_1 \mathsf{R}\, \psi_2))$*;* (iii) $\mathsf{E}^{\geq g}\psi_1 \tilde{\mathsf{U}}\, \psi_2 \equiv \mathsf{E}^{\geq g}(\psi_2 \vee \psi_1 \wedge \tilde{\mathsf{X}}\, \psi_1 \tilde{\mathsf{U}}\, \psi_2)$*;* (iv) $\mathsf{E}^{\geq g}\psi_1 \tilde{\mathsf{R}}\, \psi_2 \equiv \mathsf{E}^{\geq g}(\psi_2 \wedge (\psi_1 \vee \tilde{\mathsf{X}}\, \psi_1 \tilde{\mathsf{R}}\, \psi_2))$*;* (v) $\mathsf{A}^{<g}\psi_1 \mathsf{U}\, \psi_2 \equiv \mathsf{A}^{<g}(\psi_2 \vee \psi_1 \wedge \mathsf{X}\, \psi_1 \mathsf{U}\, \psi_2)$*;* (vi) $\mathsf{A}^{<g}\psi_1 \mathsf{R}\, \psi_2 \equiv \mathsf{A}^{<g}(\psi_2 \wedge (\psi_1 \vee \mathsf{X}\, \psi_1 \mathsf{R}\, \psi_2))$*;* (vii) $\mathsf{A}^{<g}\psi_1 \tilde{\mathsf{U}}\, \psi_2 \equiv \mathsf{A}^{<g}(\psi_2 \vee \psi_1 \wedge \tilde{\mathsf{X}}\, \psi_1 \tilde{\mathsf{U}}\, \psi_2)$*;* (viii) $\mathsf{A}^{<g}\psi_1 \tilde{\mathsf{R}}\, \psi_2 \equiv \mathsf{A}^{<g}(\psi_2 \wedge (\psi_1 \vee \tilde{\mathsf{X}}\, \psi_1 \tilde{\mathsf{R}}\, \psi_2))$.

Consider now a state formula φ on which we have to verify the equivalence between paths. Then, we may want to have that, when a world satisfies φ, all paths starting from that world are counted just once. This is because, after all, we have only one way to practically satisfy the formula.

Definition 4.2 (*State Focus*). An equivalence relation $\equiv_{\mathcal{K}}^{\cdot}$ is said to be *state focused* if and only if, given a state formula φ, if $\mathcal{K}, w \models \varphi$ then $\pi_1 \equiv_{\mathcal{K}}^{\varphi} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w)$.

THEOREM 4.2 (STATE QUANTIFICATION). *Let* $\equiv_{\cdot}$ *be a state-focused equivalence relation. Moreover, let* φ *be a state formula and* $g \in [2, \omega]$*. Then, the following holds:* (i) $\mathsf{E}^{\geq g}\varphi \equiv \mathfrak{f}$ *and* (ii) $\mathsf{A}^{<g}\varphi \equiv \mathfrak{t}$.

PROOF. Suppose by contradiction that $\mathsf{E}^{\geq g}\varphi \not\equiv \mathfrak{f}$, that is, that there is a KS $\mathcal{K}$ such that $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\varphi$, where w_0 is the initial world of $\mathcal{K}$. This means that $|(\mathrm{Pth}(\mathcal{K}, w_0, \varphi)/{\equiv_{\mathcal{K}}^{\varphi}})| \geq g$, so $\mathrm{Pth}(\mathcal{K}, w_0, \varphi) \neq \emptyset$ and then, by Item ii of Proposition 3.1, it holds that $\mathcal{K}, w_0 \models \varphi$. Now, by the state-focus property, we have that $\pi_1 \equiv_{\mathcal{K}}^{\varphi} \pi_2$, for all paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0)$. Hence, $|(\mathrm{Pth}(\mathcal{K}, w_0, \varphi)/{\equiv_{\mathcal{K}}^{\varphi}})| = 1 < g$, but this contradict the hypothesis. □

4.2. Temporal Requirements

Consider a path formula ψ. We would like that the number of paths satisfying $\mathsf{X}\, \psi$ at a world w is equal to the sum of the number of paths that satisfy ψ on all successor worlds w' of w. This requires that two paths π_1 and π_2 are distinct w.r.t. $\mathsf{X}\, \psi$ if and only if the paths $(\pi_1)_{\geq 1}$ and $(\pi_2)_{\geq 1}$ are also distinct w.r.t. ψ.

Definition 4.3 (*Next Consistency*). An equivalence relation $\equiv_{\mathcal{K}}^{\cdot}$ on paths is said to be *next consistent* if and only if it holds that $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{X}\,\psi} \pi_2$ if and only if $(\pi_1)_{\geq 1} \equiv_{\mathcal{K}}^{\psi} (\pi_2)_{\geq 1}$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w)$.

By the state focus and next-consistency properties, it is immediate to derive the following first accessory lemma.

LEMMA 4.1 (NEXT EQUIVALENCE I). *Let* $\equiv_{\mathcal{K}}^{\cdot}$ *be a state-focused and next-consistent equivalence relation. Moreover, let* $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w)$ *be two paths starting in a common world* w *and* φ *be a state formula. Then,* $(\pi_1)_1 = (\pi_2)_1 = w'$ *and* $\mathcal{K}, w' \models \varphi$ *imply* $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{X}\,\varphi} \pi_2$.

PROOF. By the state-focus property, it holds that $(\pi_1)_{\geq 1} \equiv_{\mathcal{K}}^{\varphi} (\pi_2)_{\geq 1}$. Now, by the next-consistency property, we obtain that $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{X}\,\varphi} \pi_2$. □

For a $\tilde{\mathsf{X}}\, \psi$ formula, the only difference w.r.t. $\mathsf{X}\, \psi$ is that the formula can be satisfied on a path because there are no successor worlds. In such a situation there is only one path satisfying the formula. In the other cases $\tilde{\mathsf{X}}\, \psi$ behaves just like $\mathsf{X}\, \psi$, hence, we would like the first to satisfy a similar property w.r.t. the latter. However, when ψ is a tautology, we have that $\tilde{\mathsf{X}}\, \psi$, differently from $\mathsf{X}\, \psi$, is equivalent to $\mathfrak{t}$, that is, the formula is always satisfied. For this reason all choices are indifferent and may be regarded as equivalent.

However, there may be other reasons to consider a given path formula ψ as tautological: one may consider that at w in $\mathcal{K}$, there are some "physical boundaries" on what paths starting from w can achieve. Then, it makes sense to take into account such limitations when evaluating whether a path formulas is tautological or not. For example, there may be a three-valued property, which is encoded by means of two binary variables a and b. However, the two binary variables actually encode a four-valued property. Then, it makes sense to assume, as hypothesis in the system, that one of the values, say $\neg a \wedge \neg b$, is never assumed by a Ks. In such a case, the formula $\neg(\neg a \wedge \neg b)$ can be considered equivalent to true. As a further example there may be an world w modeling an end state with a property *end* and a self loop. In such a case, the formula $\mathsf{G}\, end$ can be considered equivalent to true too.

We highlight the tautological nature of a path formula by means of a generalization of the LTL equivalence relation that depends also on the context the formulas are evaluated in, that is, on a particular Ks $\mathcal{K}$ and on one of its world w. Two path formulas should be equivalent at $\mathcal{K}, w$ only if they cannot distinguish paths in $\mathrm{Pth}(\mathcal{K}, w)$.

We now formally define the general notion of equivalence among path formulas.

Definition 4.4 (*Equivalence Structure*). An *equivalence structure* $\cong^{\cdot}_{\cdot}$ is a parametric equivalence relation among path formulas depending on a Ks $\mathcal{K}$ and one of its worlds w such that, for all path formulas ψ_1 and ψ_2, *(i)* $\psi_1 \equiv \psi_2$ implies that $\psi_1 \cong^{w}_{\mathcal{K}} \psi_2$ and *(ii)* $\psi_1 \cong^{w}_{\mathcal{K}} \psi_2$ implies that $\mathcal{K}, \pi \models \psi_1$ if and only if $\mathcal{K}, \pi \models \psi_2$, for all $\pi \in \mathrm{Pth}(\mathcal{K}, w)$.

Observe that the LTL equivalence relation $\equiv$ is a particular equivalence structure $\cong^{w}_{\mathcal{K}}$ that does not depend on the Ks $\mathcal{K}$ and world w.

At this point, we are ready to define a generic tautology.

Definition 4.5 (*Tautology Structure*). Given a Ks $\mathcal{K}$ and one of its worlds w, a path formula ψ is a $\cong^{w}_{\mathcal{K}}$-tautology if and only if $\psi \cong^{w}_{\mathcal{K}} \mathfrak{t}$.

Using this concept we can state the consistency property required by the weak next operator.

Definition 4.6 (*Weak Next Consistency*). An equivalence relation $\equiv^{\cdot}_{\mathcal{K}}$ on paths is said to be *weak next consistent* w.r.t. an equivalence structure $\cong^{\cdot}_{\mathcal{K}}$ if and only if it holds that $\pi_1 \equiv^{\tilde{\mathsf{X}}\psi}_{\mathcal{K}} \pi_2$ if and only if $\tilde{\mathsf{X}}\,\psi$ is an $\cong^{w}_{\mathcal{K}}$-tautology or $(\pi_1)_{\geq 1} \equiv^{\psi}_{\mathcal{K}} (\pi_2)_{\geq 1}$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w)$.

By the next and weak next-consistency properties, we can derive the simplification theorem for the quantifications of the weak next temporal operator.

THEOREM 4.3 (WEAK NEXT SIMPLIFICATION). *Let $\equiv^{\cdot}_{\cdot}$ be a next-consistent and weak next-consistent equivalence relation w.r.t. $\cong^{\cdot}_{\cdot}$. Moreover, let $\mathcal{K}$ be a* Ks*, ψ be a path formula and $g \in [2, \omega]$. Then, the following holds:* (i) $\mathcal{K} \models \mathsf{E}^{\geq g}\tilde{\mathsf{X}}\,\psi$ *if and only if* $\tilde{\mathsf{X}}\,\psi$ *is not an* $\cong^{w_0}_{\mathcal{K}}$*-tautology and* $\mathcal{K} \models \mathsf{E}^{\geq g}\mathsf{X}\,\psi$ *and* (ii) $\mathcal{K} \models \mathsf{A}^{<g}\mathsf{X}\,\psi$ *if and only if* $\neg\mathsf{X}\,\psi$ *is an* $\cong^{w_0}_{\mathcal{K}}$*-tautology or* $\mathcal{K} \models \mathsf{A}^{<g}\tilde{\mathsf{X}}\,\psi$*, where w_0 is the initial world of $\mathcal{K}$.*

PROOF. By hypotheses, it holds that $\pi_1 \equiv^{\tilde{\mathsf{X}}\psi}_{\mathcal{K}} \pi_2$ if and only if $\tilde{\mathsf{X}}\,\psi$ is an $\cong^{w_0}_{\mathcal{K}}$-tautology or $\pi_1 \equiv^{\mathsf{X}\psi}_{\mathcal{K}} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0)$, where w_0 is the initial world of $\mathcal{K}$.

[Only if]. If $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\tilde{\mathsf{X}}\,\psi$ then $|(\mathrm{Pth}(\mathcal{K}, w_0, \tilde{\mathsf{X}}\,\psi)/\equiv^{\tilde{\mathsf{X}}\psi}_{\mathcal{K}})| \geq g$. Since there are at least two different classes w.r.t. $\equiv^{\tilde{\mathsf{X}}\psi}_{\mathcal{K}}$ and so, at least two nonequivalent paths starting in w_0, it holds that $\tilde{\mathsf{X}}\,\psi$ cannot be an $\cong^{w_0}_{\mathcal{K}}$-tautology. Consequently, we have that $\pi_1 \equiv^{\tilde{\mathsf{X}}\psi}_{\mathcal{K}} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{X}\,\psi} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0)$. Moreover, since w_0 has necessarily a successor, by Items v and vi of Proposition 3.1, it holds that $\mathrm{Pth}(\mathcal{K}, w_0, \tilde{\mathsf{X}}\,\psi) = \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)$. Thus, we obtain that $(\mathrm{Pth}(\mathcal{K}, w_0, \tilde{\mathsf{X}}\,\psi)/\equiv_{\mathcal{K}}^{\tilde{\mathsf{X}}\psi}) = (\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)/\equiv_{\mathcal{K}}^{\mathsf{X}\psi})$. Hence, the thesis holds.

[If]. If $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\mathsf{X}\,\psi$ then $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi})| \geq g$. Since there are at least two different classes w.r.t. $\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi}$, w_0 has necessarily a successor and so, by Items v and vi of Proposition 3.1, it holds that $\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi) = \mathrm{Pth}(\mathcal{K}, w_0, \tilde{\mathsf{X}}\,\psi)$. Moreover, $\tilde{\mathsf{X}}\,\psi$ is not an $\cong_{\mathcal{K}}^{w_0}$-tautology. Consequently, we have that $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{X}\,\psi} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\tilde{\mathsf{X}}\,\psi} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0)$. Thus, we obtain that $(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi}) = (\mathrm{Pth}(\mathcal{K}, w_0, \tilde{\mathsf{X}}\,\psi)/\equiv_{\mathcal{K}}^{\tilde{\mathsf{X}}\,\psi})$. Hence, the thesis holds. □

In general, there are no GCTL* formulas expressing the fact that $\tilde{\mathsf{X}}\,\psi$ and $\neg\mathsf{X}\,\psi$ are or are not an $\cong_{\mathcal{K}}^{w_0}$-tautology. However, in the case that a particular $\cong_{\mathcal{K}}^{w_0}$-tautology of the previous formulas can be expressed with the two apposite formulas $\varphi_{\tilde{\mathsf{X}}\,\psi}$ and $\varphi_{\neg\mathsf{X}\,\psi}$, we can easily state $\mathsf{E}^{\geq g}\tilde{\mathsf{X}}\,\psi \equiv (\mathsf{E}^{\geq g}\mathsf{X}\,\psi) \wedge \neg\varphi_{\tilde{\mathsf{X}}\,\psi}$ and $\mathsf{A}^{<g}\mathsf{X}\,\psi \equiv (\mathsf{A}^{<g}\tilde{\mathsf{X}}\,\psi) \vee \varphi_{\neg\mathsf{X}\,\psi}$, for $g \in [2, \omega]$. Moreover, we recall that Items vi and xii of Proposition 3.2 assert that $\mathsf{E}^{\geq g}\tilde{\mathsf{X}}\,\psi \equiv \mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}^{\geq 1}\mathsf{X}\,\psi$ and $\mathsf{A}^{<g}\mathsf{X}\,\psi \equiv \mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t} \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\,\psi$, for $g = 1$. Then, we introduce the two macros $\mathsf{E}\tilde{\mathsf{X}}(g, \psi, \varphi)$ and $\mathsf{A}\mathsf{X}(g, \psi, \varphi)$, defined below, to represent in short the expansion formula for $\mathsf{E}\tilde{\mathsf{X}}$ and $\mathsf{A}\mathsf{X}$.

— $\mathsf{E}\tilde{\mathsf{X}}(g, \psi, \varphi) \triangleq \begin{cases} \mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}^{\geq 1}\mathsf{X}\,\psi, & \text{if } g = 1; \\ (\mathsf{E}^{\geq g}\mathsf{X}\,\psi) \wedge \varphi, & \text{otherwise.} \end{cases}$

— $\mathsf{A}\mathsf{X}(g, \psi, \varphi) \triangleq \begin{cases} \mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t} \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\,\psi, & \text{if } g = 1; \\ (\mathsf{A}^{\geq g}\tilde{\mathsf{X}}\,\psi) \vee \varphi, & \text{otherwise.} \end{cases}$

It is immediate to see that $|\mathsf{E}\tilde{\mathsf{X}}(g, \psi, \varphi)| = |\mathsf{A}\mathsf{X}(g, \psi, \varphi)| = \Theta(|\varphi| + |\psi|)$.

These properties for the next and the weak next operators allow us to say that the number of paths that satisfy $\mathsf{X}\,\psi$ or $\tilde{\mathsf{X}}\,\psi$ at world w is equal to the number of paths that satisfy ψ on some successor world w' of w. Since two paths π_1 and π_2 passing through two distinct successors may represent two different ways to satisfy $\mathsf{X}\,\psi$, we would like to consider them as distinct w.r.t. $\mathsf{X}\,\psi$. So, we should have that the two paths $(\pi_1)_{\geq 1}$ and $(\pi_2)_{\geq 1}$ are not-equivalent just because they start from different nodes. Consequently, we may want to ensure that paths starting at different successors are never counted just as one.

Definition 4.7 (*Source Dependence*). An equivalence relation $\equiv_{\mathcal{K}}^{\cdot}$ on paths is said to be *source-dependent* if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$ implies $(\pi_1)_0 = (\pi_2)_0$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K})$.

At this point, by the next-consistency and source-dependence properties it is immediate to derive the following second accessory lemma.

LEMMA 4.2 (NEXT EQUIVALENCE II). *Let $\equiv_{\mathcal{K}}^{\cdot}$ be a next-consistent and source-dependent equivalence relation. Moreover, let $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w)$ be two paths starting in a common world w. Then, $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{X}\,\psi} \pi_2$ implies $(\pi_1)_1 = (\pi_2)_1$.*

PROOF. By the next-consistency property, it holds that $(\pi_1)_{\geq 1} \equiv_{\mathcal{K}}^{\psi} (\pi_2)_{\geq 1}$. Now, by the source-dependence property, we obtain that $(\pi_1)_1 = (\pi_2)_1$. □

Before continuing with the discussion of the remaining properties, we have to make an important remark on our choice to define the semantics of GCTL* on both finite and infinite paths and, consequently, to have both the strong and weak versions of the

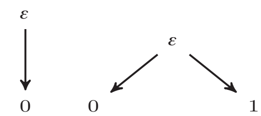


Fig. 1. The KTS $\mathcal{T}_1$ and $\mathcal{T}_2$.

temporal operators (see also Eisner et al. [2003], for further nontechnical motivations for logics over the so-called truncated paths). Suppose, for a moment, to define the GCTL* semantics only on infinite paths, that is, to consider only total KS. Under this assumption, it is immediate to see that strong and weak temporal operators are equivalent, that is, $\mathsf{X}\,\psi \equiv \tilde{\mathsf{X}}\,\psi$, $\psi_1 \mathsf{U}\,\psi_2 \equiv \psi_1 \tilde{\mathsf{U}}\,\psi_2$, and $\psi_1 \mathsf{R}\,\psi_2 \equiv \psi_1 \tilde{\mathsf{R}}\,\psi_2$. In particular, it holds that $\mathsf{X}\,\mathfrak{t} \equiv \mathfrak{t}$ and so, for the syntax-independence and state-focus (specifically, here we need only that all paths are equivalent w.r.t. $\mathfrak{t}$) properties, we obtain that $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{X}\,\mathfrak{t}} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K})$. Hence, if we want to preserve the syntax independence, we are not able to simply count the number of successors of a given world, by using the formula $\mathsf{E}^{\geq g}\mathsf{X}\,\mathfrak{t}$, without asserting any stronger property. However, all the classical graded logics, such as the GμCALCULUS, allow such a counting. Moreover, consider two paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w)$ such that $(\pi_1)_1 \neq (\pi_2)_1$. By the previous lemma, we have that $\pi_1 \not\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathfrak{t}} \pi_2$, reaching in this way a contradiction. Hence, it is evident that it is impossible to cast together the three properties of syntax independence, next consistency, and source dependence in the framework of logics on infinite paths only. If we want to restrict ourselves to such a framework, we have to drop at least one property between the last two, changing completely the semantics of the logic and indirectly the interesting relationship with the GμCALCULUS shown in the next section. We can now return to the main track of thought of this section. In particular, we can enunciate a fundamental result on the loss of the bisimulation invariance, since the operation of counting is not bisimilar invariant at all, and, consequently, on the more expressiveness of the graded w.r.t. the related ungraded logics.

THEOREM 4.4 (BISIMILARITY VARIANCE). *Let* $\equiv^{\cdot}_{\cdot}$ *be a next-consistent and source-dependent equivalence relation. Then* GCTL *and* GCTL* *are not invariant under bisimilarity. Moreover, they are more expressive than* CTL *and* CTL*, *respectively.*

PROOF. We show that GCTL distinguishes between bisimilar models. Consider the two KTS $\mathcal{T}_1$ and $\mathcal{T}_2$ such as $\mathcal{T}_1$ contains only the root and one successor, while $\mathcal{T}_2$ contains also another successor of the root (see Figure 1). Formally, $\mathcal{T}_1 = \langle \mathrm{AP}, \mathrm{W}_1, R_1, \mathsf{L}_1, \varepsilon \rangle$, with $\mathrm{AP} = \emptyset$, $\mathrm{W}_1 = \{\varepsilon, 0\}$, and $R_1 = \{(\varepsilon, 0)\}$, and $\mathcal{T}_2 = \langle \mathrm{AP}, \mathrm{W}_2, R_2, \mathsf{L}_2, \varepsilon \rangle$, with $\mathrm{W}_2 = \mathrm{W}_1 \cup \{1\}$, and $R_2 = R_1 \cup \{(\varepsilon, 1)\}$. By the definition of bisimilarity, it is immediate to see that $\mathcal{T}_1$ and $\mathcal{T}_2$ are bisimilar. Now, consider the formula $\varphi = \mathsf{E}^{\geq 2}\mathsf{X}\,\mathfrak{t}$. It is evident that $\mathrm{Pth}(\mathcal{T}_1, \varepsilon, \mathsf{X}\,\mathfrak{t}) = \{\pi_1\}$ with $\pi_1 = \varepsilon \cdot 0$, so $|(\mathrm{Pth}(\mathcal{T}_1, \varepsilon, \mathsf{X}\,\mathfrak{t})/\equiv_{\mathcal{T}_1}^{\mathsf{X}\,\mathfrak{t}})| = 1$ and then $\mathcal{T}_1 \not\models \varphi$. On the contrary, $\mathrm{Pth}(\mathcal{T}_2, \varepsilon, \mathsf{X}\,\mathfrak{t}) = \{\pi_1, \pi_2\}$ with $\pi_2 = \varepsilon \cdot 1$. Since $(\pi_1)_1 \neq (\pi_2)_2$, by Lemma 4.2, we have that $\pi_1 \not\equiv_{\mathcal{T}_2}^{\mathsf{X}\,\mathfrak{t}} \pi_2$, so $|(\mathrm{Pth}(\mathcal{T}_2, \varepsilon, \mathsf{X}\,\mathfrak{t})/\equiv_{\mathcal{T}_2}^{\mathsf{X}\,\mathfrak{t}})| = 2$ and then $\mathcal{T}_2 \models \varphi$. Hence, φ is not an invariant for the two KTS $\mathcal{T}_1$ and $\mathcal{T}_2$ and so, it can distinguish between bisimilar models. Now, it is known that both CTL and CTL* are invariant under bisimulation, so, they cannot distinguish between $\mathcal{T}_1$ and $\mathcal{T}_2$. Moreover, CTL and CTL* are sublogics of GCTL and GCTL*, respectively. Thus, we have that the latter can characterize more models than those characterizable by the former logic. Consequently, the theses hold. □

As third and last accessory lemma, we derive an important and completely general combinatorial property on the dimension of groupings of equivalence classes depending on their size.

LEMMA 4.3 (CLASSES COUNTING). *Let* $\equiv$ *be an equivalence relation on a finite set* S*. Moreover, let* $\mathrm{M}_n = \{\mathrm{D} \in (\mathrm{S}/\equiv) : |\mathrm{D}| = n\}$ *be the set of equivalence classes w.r.t.* $\equiv$ *having size n, for each* $n \in [1, |\mathrm{S}|]$*. Then, there is a partition solution* $p \in \mathrm{P}(|\mathrm{S}|)$ *such that* $|\mathrm{M}_n| = (p)_n$*, for each* $n \in [1, |\mathrm{S}|]$.

PROOF. First note that, by definition, $\mathrm{M}_{n_1} \cap \mathrm{M}_{n_2} = \emptyset$, for all $n_1, n_2 \in [1, |\mathrm{S}|]$ with $n_1 \neq n_2$. Moreover, for all $\mathrm{D}_1, \mathrm{D}_2 \in \mathrm{M}_n$ with $\mathrm{D}_1 \neq \mathrm{D}_2$, it holds that $\mathrm{D}_1 \cap \mathrm{D}_2 = \emptyset$, since they are different equivalence classes. Furthermore, it is evident that $\mathrm{S} = \bigcup_{n=1}^{|\mathrm{S}|} \bigcup_{\mathrm{D} \in \mathrm{M}_n} \mathrm{D}$. So, we have that $|\mathrm{S}| = |\bigcup_{n=1}^{|\mathrm{S}|} \bigcup_{\mathrm{D} \in \mathrm{M}_n} \mathrm{D}| = \sum_{n=1}^{|\mathrm{S}|} \sum_{\mathrm{D} \in \mathrm{M}_n} |\mathrm{D}| = \sum_{n=1}^{|\mathrm{S}|} \sum_{\mathrm{D} \in \mathrm{M}_n} n = \sum_{n=1}^{|\mathrm{S}|} n \cdot |\mathrm{M}_n|$. Hence, by the definition of partition solution, the thesis holds. $\square$

Finally, we can enunciate two theorems that generalize to the case of graded quantifiers the classical CTL* expansion equivalences $\mathsf{EX}\,\psi \equiv \mathsf{EX}\,\mathsf{E}\psi$ and $\mathsf{A}\tilde{\mathsf{X}}\,\psi \equiv \mathsf{A}\tilde{\mathsf{X}}\,\mathsf{A}\psi$. The first property is of crucial importance for the characterization of GCTL, without quantifiers with infinite degrees (i.e., without $\mathsf{E}^{\geq\omega}\psi$ and $\mathsf{A}^{<\omega}\psi$), as a fragment of the GμCALCULUS, as showed in the next section.

THEOREM 4.5 (NEXT EXPANSION I). *Let* $\equiv_{\cdot}$ *be a state-focused, next-consistent, and source-dependent equivalence relation. Moreover, let* ψ *be a path formula and* $g \in [1, \omega[$*. Then, the following equivalences hold:* (i) $\mathsf{E}^{\geq g}\mathsf{X}\,\psi \equiv \bigvee_{c \in \mathrm{C}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq (c)_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$ *and* (ii) $\mathsf{A}^{<g}\tilde{\mathsf{X}}\,\psi \equiv \bigvee_{c \in \mathrm{C}(g-1)} \bigwedge_{i=1}^{g} \mathsf{A}^{\leq (c)_i}\tilde{\mathsf{X}}\,\mathsf{A}^{<i}\psi$.

PROOF. [Only if]. If $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\mathsf{X}\,\psi$ then $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi})| \geq g$, where $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, w_0\rangle$. Thus, there is a set $\mathrm{S} \subseteq \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)$ of g nonequivalent paths w.r.t. $\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi}$. Each path in S is a representative of a different class, so $|\mathrm{S}| = |(\mathrm{S}/\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi})| = g$.

Now let $\overset{succ}{\equiv}$ be the equivalence relation on $\mathrm{Pth}(\mathcal{K})$ such that $\pi_1 \overset{succ}{\equiv} \pi_2$ if and only if $(\pi_1)_1 = (\pi_2)_1$. Moreover, let $\mathrm{M}_n \triangleq \{\mathrm{D} \in (\mathrm{S}/\overset{succ}{\equiv}) : |\mathrm{D}| = n\}$ be the set of equivalence classes w.r.t. $\overset{succ}{\equiv}$ having size $n \in [1, g]$. By Lemma 4.3, there is a partition solution $p \in \mathrm{P}(g)$ such that $|\mathrm{M}_n| = (p)_n$, for all $n \in [1, g]$. At this point, we can write $\mathrm{M}_n = \{\mathrm{D}_{n,1}, \ldots, \mathrm{D}_{n,(p)_n}\}$. Furthermore, we can associate to each class $\mathrm{D}_{n,j}$ a different successor $w_{n,j}$ of the initial world w_0 such that $w_{n,j} = (\pi)_1$, for all $\pi \in \mathrm{D}_{n,j}$.

Since $\mathrm{D}_{n,j} \subseteq \mathrm{S}$, we have that $\mathcal{K}, \pi \models \mathsf{X}\,\psi$ and so, by Item v of Proposition 3.1, $\mathcal{K}, \pi_{\geq 1} \models \psi$, for all $\pi \in \mathrm{D}_{n,j}$. Hence, let $\mathrm{D}'_{n,j} \triangleq \{\pi_{\geq 1} : \pi \in \mathrm{D}_{n,j}\}$, we obtain that $\mathrm{D}'_{n,j} \subseteq \mathrm{Pth}(\mathcal{K}, w_{n,j}, \psi)$. Note that $|\mathrm{D}'_{n,j}| = |\mathrm{D}_{n,j}| = n$. Moreover, by the next-consistency property, since $\pi_1 \not\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{D}_{n,j}$ with $\pi_1 \neq \pi_2$, we obtain that $(\pi_1)_{\geq 1} \not\equiv_{\mathcal{K}}^{\psi} (\pi_2)_{\geq 1}$ and so $|(\mathrm{D}'_{n,j}/\equiv_{\mathcal{K}}^{\psi})| = |\mathrm{D}'_{n,j}| = n$. Thus, we have that $|(\mathrm{Pth}(\mathcal{K}, w_{n,j}, \psi)/\equiv_{\mathcal{K}}^{\psi})| \geq n$. Hence, $\mathcal{K}, w_{n,j} \models \mathsf{E}^{\geq i}\psi$, for all $i \in [1, n]$. By Items ii and v of Proposition 3.1, the last statement implies that $\mathcal{K}, \pi \models \mathsf{X}\,\mathsf{E}^{\geq i}\psi$, for all $\pi \in \mathrm{D}_{n,j}$ with $n \in [i, g]$ and $j \in [1, (p)_n]$.

By Lemma 4.1, it holds that $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{D}_{n,j}$, and thus $|(\mathrm{D}_{n,j}/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi})| = 1$. On the contrary, by Lemma 4.2, for all $\pi_1 \in \mathrm{D}_{n_1,j_1}$ and $\pi_2 \in \mathrm{D}_{n_2,j_2}$ with $n_1 \neq n_2$ or $j_1 \neq j_2$, since $(\pi_1)_1 = w_{n_1,j_1} \neq w_{n_2,j_2} = (\pi_2)_1$, it holds that $\pi_1 \not\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi} \pi_2$ and thus $((\mathrm{D}_{n_1,j_1} \cup \mathrm{D}_{n_2,j_2})/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi}) = (\mathrm{D}_{n_1,j_1}/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi}) \cup (\mathrm{D}_{n_2,j_2}/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi})$.

Now, we can estimate the number of equivalence classes w.r.t. $\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi}$ of the set of paths $\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\mathsf{E}^{\geq i}\psi)$. Since, as previously proved, $\bigcup_{n=i}^{g} \bigcup_{j=1}^{(p)_n} \mathrm{D}_{n,j} \subseteq \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\mathsf{E}^{\geq i}\psi)$, we have that $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\mathsf{E}^{\geq i}\psi)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi})| \geq |((\bigcup_{n=i}^{g} \bigcup_{j=1}^{(p)_n} \mathrm{D}_{n,j})/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi})| = |\bigcup_{n=i}^{g} \bigcup_{j=1}^{(p)_n} (\mathrm{D}_{n,j}/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi})| = \sum_{n=i}^{g} \sum_{j=1}^{(p)_n} |(\mathrm{D}_{n,j}/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq i}\psi})| = \sum_{n=i}^{g} \sum_{j=1}^{(p)_n} 1 = \sum_{n=i}^{g} (p)_n$.

Now let $c \in \mathbb{N}^n$ be the vector such that $(c)_i = \sum_{n=i}^{g}(p)_n$. At this point, it is immediate to see that $\mathcal{K}, w_0 \models \mathsf{E}^{\geq (c)_i} \mathsf{X}\, \mathsf{E}^{\geq i} \psi$. Since the previous reasoning can be done for every $i \in [1, g]$, we also have $\mathcal{K}, w_0 \models \bigwedge_{i=1}^{g} \mathsf{E}^{\geq (c)_i} \mathsf{X}\, \mathsf{E}^{\geq i} \psi$. Now, by definition of cumulative-partition solution, we have that $c \in \mathrm{C}(g)$. So, $\mathcal{K}, w_0 \models \bigvee_{c \in \mathrm{C}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq (c)_i} \mathsf{X}\, \mathsf{E}^{\geq i} \psi$.

[If]. If $\mathcal{K}, w_0 \models \bigvee_{c \in \mathrm{C}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq (c)_i} \mathsf{X}\, \mathsf{E}^{\geq i} \psi$ then there is a cumulative-partition solution $c \in \mathrm{C}(g)$ such that, for all $i \in [1, g]$, it holds that $\mathcal{K}, w_0 \models \mathsf{E}^{\geq (c)_i} \mathsf{X}\, \mathsf{E}^{\geq i} \psi$ and so $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\, \mathsf{E}^{\geq i} \psi)/{\equiv_{\mathcal{K}}^{\mathsf{X}\, \mathsf{E}^{\geq i} \psi}})| \geq (c)_i$, where $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, w_0 \rangle$. Now let $p \in \mathbb{N}^n$ be a vector such that $(p)_g = (c)_g$ and $(p)_i = (c)_i - (c)_{i+1}$, for all $i \in [1, g[$. By definition of cumulative-partition solution, it is immediate to see that p is a partition solution, that is, $p \in \mathrm{P}(g)$.

First note that the set $\mathrm{V}_i \triangleq \{w \in \mathrm{W} : (w_0, w) \in R \wedge \mathcal{K}, w \models \mathsf{E}^{\geq i} \psi\}$ of successors of the initial world w_0 satisfying $\mathsf{E}^{\geq i} \psi$ has cardinality greater than or equal to $(c)_i$. Indeed, let $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\, \mathsf{E}^{\geq i} \psi)$ be two paths such that $\pi_1 \not\equiv_{\mathcal{K}}^{\mathsf{X}\, \mathsf{E}^{\geq i} \psi} \pi_2$. Then, by Lemma 4.1, we have that $(\pi_1)_1 \neq (\pi_2)_1$. So, since, as shown before, there exist at least $(c)_i$ nonequivalent paths w.r.t. $\equiv_{\mathcal{K}}^{\mathsf{X}\, \mathsf{E}^{\geq i} \psi}$, we obtain that there are at least $(c)_i$ different successors of w_0.

Now, for each $i \in [1, g[$, let $\mathrm{U}_i \subseteq \mathrm{V}_i$ be a set of $(p)_i$ worlds such that $\mathrm{U}_i \cap \mathrm{U}_j = \emptyset$, for all $j \in]i, g]$. By finite induction, it is immediate to see that we can effectively construct such sets, since $|\mathrm{V}_i \setminus \bigcup_{j=i+1}^{g} \mathrm{U}_j| \geq (c)_i - \sum_{j=i+1}^{g} |\mathrm{U}_j| = (c)_i - \sum_{j=i+1}^{g}(p)_j = (c)_i - (c)_{i+1} = (p)_i$. At this point, we can write $\mathrm{U}_i = \{w_{i,1}, \ldots, w_{i,(p)_i}\}$. Furthermore, since $\mathcal{K}, w_{i,j} \models \mathsf{E}^{\geq i} \psi$, we can associate to each world $w_{i,j}$ a set $\mathrm{D}'_{i,j} \subseteq \mathrm{Pth}(\mathcal{K}, w_{i,j}, \psi)$ of i nonequivalent paths w.r.t. $\equiv_{\mathcal{K}}^{\psi}$. Now, let $\mathrm{D}_{i,j} \triangleq \{\pi \in \mathrm{Pth}(\mathcal{K}, w_0) : \pi_{\geq 1} \in \mathrm{D}'_{i,j}\}$. By Item v of Proposition 3.1, $\mathrm{D}_{i,j} \subseteq \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\, \psi)$. Note that $|\mathrm{D}_{i,j}| = |\mathrm{D}'_{i,j}| = i$. By the next-consistency property, since $(\pi_1)_{\geq 1} \not\equiv_{\mathcal{K}}^{\psi} (\pi_2)_{\geq 1}$, for all $\pi_1, \pi_2 \in \mathrm{D}_{n,j}$ with $\pi_1 \neq \pi_2$, we obtain that $\pi_1 \not\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi} \pi_2$ and so $|(\mathrm{D}_{i,j}/{\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}})| = |\mathrm{D}_{i,j}| = i$. Moreover, by Lemma 4.2, for all $\pi_1 \in \mathrm{D}_{i_1,j_1}$ and $\pi_2 \in \mathrm{D}_{i_2,j_2}$ with $i_1 \neq i_2$ or $j_1 \neq j_2$, since $(\pi_1)_1 = w_{i_1,j_1} \neq w_{i_2,j_2} = (\pi_2)_1$, it holds that $\pi_1 \not\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi} \pi_2$ and thus $((\mathrm{D}_{i_1,j_1} \cup \mathrm{D}_{i_2,j_2})/{\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}}) = (\mathrm{D}_{i_1,j_1}/{\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}}) \cup ((\mathrm{D}_{i_2,j_2}/{\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}}))$.

Now, we can estimate the number of equivalence classes w.r.t. $\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}$ of the set of paths $\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\, \psi)$. Since, as previously proved, $\bigcup_{i=1}^{g} \bigcup_{j=1}^{(p)_i} \mathrm{D}_{i,j} \subseteq \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\, \psi)$, we have that $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\, \psi)/{\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}})| \geq |((\bigcup_{i=1}^{g} \bigcup_{j=1}^{(p)_i} \mathrm{D}_{i,j})/{\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}})| = |\bigcup_{i=1}^{g} \bigcup_{j=1}^{(p)_i} (\mathrm{D}_{i,j}/{\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}})| = \sum_{i=1}^{g} \sum_{j=1}^{(p)_i} |(\mathrm{D}_{i,j}/{\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}})| = \sum_{i=1}^{g} \sum_{j=1}^{(p)_i} i = \sum_{i=1}^{g} i \cdot (p)_i = g$. The last equality is due to the fact that p is a partition solution. Hence, we have that $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g} \mathsf{X}\, \psi$. □

THEOREM 4.6 (NEXT EXPANSION II). *Let $\equiv_{\cdot}^{\cdot}$ be a state-focused, next-consistent, and source-dependent equivalence relation. Moreover, let ψ be a path formula. Then, the following equivalences hold:* (i) $\mathsf{E}^{\geq \omega} \mathsf{X}\, \psi \equiv \mathsf{E}^{\geq \omega} \mathsf{X}\, \mathsf{E}^{\geq 1} \psi \vee \mathsf{E}^{\geq 1} \mathsf{X}\, \mathsf{E}^{\geq \omega} \psi$ *and* (ii) $\mathsf{A}^{< \omega} \tilde{\mathsf{X}}\, \psi \equiv \mathsf{A}^{< \omega} \tilde{\mathsf{X}}\, \mathsf{A}^{< 1} \psi \wedge \mathsf{A}^{< 1} \tilde{\mathsf{X}}\, \mathsf{A}^{< \omega} \psi$.

PROOF. [Only if]. If $\mathcal{K}, w_0 \models \mathsf{E}^{\geq \omega} \mathsf{X}\, \psi$ then $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\, \psi)/{\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}})| \geq \omega$, where $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, w_0 \rangle$. Thus, there is an infinite set $\mathrm{S} \subseteq \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\, \psi)$ of nonequivalent paths w.r.t. $\equiv_{\mathcal{K}}^{\mathsf{X}\, \psi}$.

Now let $\overset{succ}{\equiv}$ be the equivalence relation on $\mathrm{Pth}(\mathcal{K})$ such that $\pi_1 \overset{succ}{\equiv} \pi_2$ if and only if $(\pi_1)_1 = (\pi_2)_1$. Moreover, let $\mathrm{M} \triangleq (\mathrm{S}/\overset{succ}{\equiv})$. To each class $\mathrm{D} \in \mathrm{M}$ we can associate a different successor w_{D} of the initial world w_0 such that $w_{\mathrm{D}} = (\pi)_1$, for all $\pi \in \mathrm{D}$.

Since $\mathrm{D} \subseteq \mathrm{S}$, we have that $\mathcal{K}, \pi \models \mathsf{X}\, \psi$ and so, by Item v of Proposition 3.1, $\mathcal{K}, \pi_{\geq 1} \models \psi$, for all $\pi \in \mathrm{D}$. Hence, let $\mathrm{D}' \triangleq \{\pi_{\geq 1} : \pi \in \mathrm{D}\}$, we obtain that $\mathrm{D}' \subseteq \mathrm{Pth}(\mathcal{K}, w_{n,j}, \psi)$. Note that $|D'| = |D|$. Moreover, by the next-consistency property, since $\pi_1 \not\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi} \pi_2$, for all $\pi_1, \pi_2 \in D$ with $\pi_1 \neq \pi_2$, we obtain that $(\pi_1)_{\geq 1} \not\equiv_{\mathcal{K}}^{\psi} (\pi_2)_{\geq 1}$ and so $|(D'/\equiv_{\mathcal{K}}^{\psi})| = |D'|$. Consequently, it holds that $|(\mathrm{Pth}(\mathcal{K}, w_D, \psi)/\equiv_{\mathcal{K}}^{\psi})| \geq |D|$. Thus, $\mathcal{K}, w_D \models \mathsf{E}^{\geq |D|}\psi$. The last statement implies that $\mathcal{K}, \pi \models \mathsf{X}\,\mathsf{E}^{\geq |D|}\psi$, for all $\pi \in D$.

At this point, we have two possibilities, each implying the truth of one of the two disjuncts in the formula $\mathsf{E}^{\geq\omega}\mathsf{X}\,\mathsf{E}^{\geq 1}\psi \vee \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq\omega}\psi$: either $|M| = \omega$ or $|M| < \omega$.

In the first case, each class $D \in M$ may be finite, so we can assert at most that $|D| \geq 1$, which implies $\mathcal{K}, \pi \models \mathsf{X}\,\mathsf{E}^{\geq 1}\psi$, for all $\pi \in D$. By Lemma 4.2, for all $\pi_1 \in D_1$ and $\pi_2 \in D_2$ with $D_1 \neq D_2$, since $(\pi_1)_1 = w_{D_1} \neq w_{D_2} = (\pi_2)_1$, it holds that $\pi_1 \not\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi} \pi_2$ and thus $((D_1 \cup D_2)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi}) = (D_1/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi}) \cup (D_2/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi})$. Now, since $\bigcup_{D \in M} D \subseteq \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\mathsf{E}^{\geq 1}\psi)$, we have that $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\mathsf{E}^{\geq 1}\psi)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi})| \geq |((\bigcup_{D \in M} D)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi})| = |\bigcup_{D \in M}(D/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi})| = \sum_{D \in M} |(D/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi})| \geq \sum_{D \in M} 1 = |M| = \omega$. Hence, $\mathcal{K}, w_0 \models \mathsf{E}^{\geq\omega}\mathsf{X}\,\mathsf{E}^{\geq 1}\psi$.

In the second case, since $S = \bigcup_{D \in M} D$ and so $|S| = \sum_{D \in M} |D|$, we have that there is a class $D \in M$ such that $|D| = \omega$. Thus, $\mathcal{K}, \pi \models \mathsf{X}\,\mathsf{E}^{\geq\omega}\psi$, for all $\pi \in D$. This implies that $|\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\mathsf{E}^{\geq\omega}\psi)| \geq 1$ and so $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\mathsf{E}^{\geq\omega}\psi)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq\omega}\psi})| \geq 1$, which means that $\mathcal{K}, w_0 \models \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq\omega}\psi$.

[If]. On one hand, if $\mathcal{K}, w_0 \models \mathsf{E}^{\geq\omega}\mathsf{X}\,\mathsf{E}^{\geq 1}\psi$, then $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\mathsf{E}^{\geq 1}\psi)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi})| \geq \omega$, where $\mathcal{K} = \langle AP, W, R, \mathsf{L}, w_0 \rangle$. Now, let $V \triangleq \{w \in W : (w_0, w) \in R \wedge \mathcal{K}, w \models \mathsf{E}^{\geq 1}\psi\}$ be the set of successors of the initial world w_0 satisfying $\mathsf{E}^{\geq 1}\psi$. It is immediate to see that $|V| = \omega$. Indeed, let $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\mathsf{E}^{\geq 1}\psi)$ be two paths such that $\pi_1 \not\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi} \pi_2$. Then, by Lemma 4.1, we have that $(\pi_1)_1 \neq (\pi_2)_1$. So, since there exist infinite non-equivalent paths w.r.t. $\equiv_{\mathcal{K}}^{\mathsf{X}\,\mathsf{E}^{\geq 1}\psi}$, we obtain that there are infinite different successors of w_0. At this point, by Item v of Proposition 3.1, we can associate a path $\pi_w \in \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)$ with $(\pi_w)_1 = w$ to each world $w \in V$. Let $D \triangleq \{\pi_w : w \in V\}$ be the set of all such paths. It is evident that $|D| = |V| = \omega$. Now, by Lemma 4.2, for all $\pi_{w_1}, \pi_{w_2} \in D$ with $w_1 \neq w_2$, it holds that $\pi_{w_1} \not\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi} \pi_{w_2}$ and thus $|(D/\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi})| = |D|$. Since $D \subseteq \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)$, we have that $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi})| \geq |(D/\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi})| = |D| = \omega$. Hence, $\mathcal{K}, w_0 \models \mathsf{E}^{\geq\omega}\mathsf{X}\,\psi$.

On the other hand, if $\mathcal{K}, w_0 \models \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq\omega}\psi$, by Items ii and v of Proposition 3.1, there is a successor $w \in W$ with $(w_0, w) \in R$ of the initial world w_0 satisfying $\mathsf{E}^{\geq\omega}\psi$. Hence, $|(\mathrm{Pth}(\mathcal{K}, w, \psi)/\equiv_{\mathcal{K}}^{\psi})| \geq \omega$. Moreover, let $D' \subseteq \mathrm{Pth}(\mathcal{K}, w, \psi)$ be a set of infinite of non-equivalent paths w.r.t. $\equiv_{\mathcal{K}}^{\psi}$ and $D \triangleq \{\pi \in \mathrm{Pth}(\mathcal{K}, w_0) : \pi_{\geq 1} \in D'\}$ be the set of their extensions with w_0. It is evident that $|D| = |D'| = \omega$. By the next-consistency property, since $(\pi_1)_{\geq 1} \not\equiv_{\mathcal{K}}^{\psi} (\pi_2)_{\geq 1}$, for all $\pi_1, \pi_2 \in D$ with $\pi_1 \neq \pi_2$, we obtain that $\pi_1 \not\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi} \pi_2$ and so $|(D/\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi})| = |D|$. Now, by Item v of Proposition 3.1, $D \subseteq \mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)$. Thus, we have that $|(\mathrm{Pth}(\mathcal{K}, w_0, \mathsf{X}\,\psi)/\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi})| \geq |(D/\equiv_{\mathcal{K}}^{\mathsf{X}\,\psi})| = |D| = \omega$. Hence, $\mathcal{K}, w_0 \models \mathsf{E}^{\geq\omega}\mathsf{X}\,\psi$. □

In the following, we use the four expressions $\mathsf{EX}(g, \psi)$, $\mathsf{A}\tilde{\mathsf{X}}(g, \psi)$, $\mathsf{EX}'(g, \psi)$, and $\mathsf{A}\tilde{\mathsf{X}}'(g, \psi)$ defined below to represent in short the expansion formulas for the X and $\tilde{\mathsf{X}}$ temporal operators derived in the previous two theorems.

— $$\mathsf{EX}(g, \psi) \triangleq \begin{cases} \bigvee_{c \in C(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq (c)_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi, & \text{if } g < \omega; \\ \mathsf{E}^{\geq\omega}\mathsf{X}\,\mathsf{E}^{\geq 1}\psi \vee \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq\omega}\psi, & \text{otherwise.} \end{cases}$$

— $$\mathsf{A}\tilde{\mathsf{X}}(g, \psi) \triangleq \begin{cases} \bigvee_{c \in C(g-1)} \bigwedge_{i=1}^{g} \mathsf{A}^{\leq (c)_i}\tilde{\mathsf{X}}\,\mathsf{A}^{< i}\psi, & \text{if } g < \omega; \\ \mathsf{A}^{<\omega}\mathsf{X}\,\mathsf{A}^{<1}\psi \wedge \mathsf{A}^{<1}\mathsf{X}\,\mathsf{A}^{<\omega}\psi, & \text{otherwise.} \end{cases}$$

—$\mathsf{E\acute{X}}(g, \psi) \triangleq \begin{cases} \bigvee_{c \in \mathrm{C}(g)}^{(c)_g = 0} \bigwedge_{i=1}^{g-1} \mathsf{E}^{\geq (c)_i} \mathsf{X}\, \mathsf{E}^{\geq i} \psi, & \text{if } g < \omega; \\ \mathsf{E}^{\geq \omega} \mathsf{X}\, \mathsf{E}^{\geq 1} \psi, & \text{otherwise.} \end{cases}$

—$\mathsf{A\acute{\tilde{X}}}(g, \psi) \triangleq \begin{cases} \bigvee_{c \in \mathrm{C}(g-1)} \bigwedge_{i=1}^{g-1} \mathsf{A}^{\leq (c)_i} \tilde{\mathsf{X}}\, \mathsf{A}^{< i} \psi, & \text{if } g < \omega; \\ \mathsf{A}^{< \omega} \mathsf{X}\, \mathsf{A}^{< 1} \psi, & \text{otherwise.} \end{cases}$

In this way, we obtain that $\mathsf{E}^{\geq g} \mathsf{X}\, \psi \equiv \mathsf{EX}(g, \psi) \equiv \mathsf{E\acute{X}}(g, \psi) \vee \mathsf{E}^{\geq 1} \mathsf{X}\, \mathsf{E}^{\geq g} \psi$ and $\mathsf{A}^{< g} \tilde{\mathsf{X}}\, \psi \equiv \mathsf{A}\tilde{\mathsf{X}}(g, \psi) \equiv \mathsf{A\acute{\tilde{X}}}(g, \psi) \wedge \mathsf{A}^{< 1} \tilde{\mathsf{X}}\, \mathsf{A}^{< g} \psi$, for all $g \in \widehat{\mathbb{N}}$. For the existential case, the second equivalence for finite degree is due to the fact that, when $(c)_g = 1$, it holds that $\bigwedge_{i=1}^{g} \mathsf{E}^{\geq (c)_i} \mathsf{X}\, \mathsf{E}^{\geq i} \psi = \bigwedge_{i=1}^{g} \mathsf{E}^{\geq 1} \mathsf{X}\, \mathsf{E}^{\geq i} \psi \equiv \mathsf{E}^{\geq 1} \mathsf{X}\, \mathsf{E}^{\geq g} \psi$. For the universal case, instead, the same equivalence is derived by the observation that, since $(c)_g = 0$, each disjunct necessarily contains the conjunct $\mathsf{A}^{\leq 0} \tilde{\mathsf{X}}\, \mathsf{A}^{< g} \psi$.

Now, it is interesting to note that, for finite degrees, the formula $\mathsf{EX}(g, \psi)$ allows to partition at least g paths through $c_1 \leq g$ successor worlds, for a given vector $c \in \mathrm{C}(g)$. Indeed, c_i is the number of successor worlds from which at least i paths satisfying ψ start. Therefore, c_1 is a sufficient bound on the number of successor worlds we have to consider to ensure the satisfiability of the formula. A similar dual reasoning can be done for the universal formula $\mathsf{A}\tilde{\mathsf{X}}(g, \psi)$.

Observe that $\mathsf{EX}(1, \psi)$ and $\mathsf{A}\tilde{\mathsf{X}}(1, \psi)$ are equal to the classical CTL* expansions $\mathsf{EX}\, \mathsf{E}\psi$ and $\mathsf{A}\tilde{\mathsf{X}}\, \mathsf{A}\psi$, respectively.

By a simple calculation, it follows that $(g-1) \cdot (|\mathrm{C}(g)| - 1) \cdot (|\psi| + 4) - 1 = |\mathsf{E\acute{X}}(g, \psi)| < |\mathsf{EX}(g, \psi)| = g \cdot |\mathrm{C}(g)| \cdot (|\psi| + 4) - 1$ and $(g-1) \cdot |\mathrm{C}(g-1)| \cdot (|\psi| + 4) - 1 = |\mathsf{A\acute{\tilde{X}}}(g, \psi)| < |\mathsf{A}\tilde{\mathsf{X}}(g, \psi)| = g \cdot |\mathrm{C}(g-1)| \cdot (|\psi| + 4) - 1$. So, both the lengths of $\mathsf{EX}(g, \psi)$ and $\mathsf{E\acute{X}}(g, \psi)$ are $\Theta((|\psi| + 4) \cdot 2^{k \cdot \sqrt{g}})$, while those of $\mathsf{A}\tilde{\mathsf{X}}(g, \psi)$ and $\mathsf{A\acute{\tilde{X}}}(g, \psi)$ are $\Theta((|\psi| + 4) \cdot 2^{k \cdot \sqrt{g-1}})$, for a constant k. Furthermore, the degree of $\mathsf{EX}(g, \psi)$, $\mathsf{A}\tilde{\mathsf{X}}(g, \psi)$, $\mathsf{E\acute{X}}(g, \psi)$, and $\mathsf{A\acute{\tilde{X}}}(g, \psi)$ is $\max\{g, \mathring{\psi}\}$. As an example, consider the formula $\varphi = \mathsf{E}^{\geq g} \mathsf{X}\, \mathsf{X}\, p$. It is evident that $|\varphi| = 4$, $\mathring{\varphi} = g$, and $\|\varphi\| = 4 + \lceil \log(g) \rceil$. Moreover, $|\mathsf{EX}(g, \mathsf{X}\, p)| = \Theta(2^{k \cdot \sqrt{g}}) = \Theta(2^{k \cdot \sqrt{2^{\|\varphi\| - 4}}})$. Hence, the length of an expansion $\mathsf{EX}(g, \psi)$ can be, in general, double exponential in the size of the original formula, also in the case its length is constant. The same thing happens for the expansion $\mathsf{A}\tilde{\mathsf{X}}(g, \psi)$.

4.3. Boolean Requirements

At this point, we can reason about the properties that an equivalence has to satisfy w.r.t. the positive Boolean combination of formulas.

Suppose we have two path formulas ψ_1 and ψ_2. We would like to have that, from a given world, both the number of paths that satisfy ψ_1 and ψ_2 are not less than those satisfying their conjunction. Hence, we need that paths equivalent w.r.t. both ψ_1 and ψ_2 are equivalent w.r.t. $\psi_1 \wedge \psi_2$ too, otherwise, each equivalence class for ψ_1 and ψ_2 may provide more than one equivalence class for $\psi_1 \wedge \psi_2$ allowing the latter formula to have more paths. Moreover, we would like that, among the paths that satisfy ψ_1 (resp., ψ_2), the number of those satisfying ψ_2 (resp., ψ_1) is equal to those satisfying $\psi_1 \wedge \psi_2$. Hence, we need that paths equivalent w.r.t. $\psi_1 \wedge \psi_2$ are also equivalent w.r.t. both ψ_1 and ψ_2.

Definition 4.8 (*Conjunction Consistency*). An equivalence relation $\equiv_{\mathcal{K}}^{\cdot}$ on paths is said to be *conjunction consistent* if and only if it holds that $\pi_1 \equiv_{\mathcal{K}}^{\psi_1 \wedge \psi_2} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi_1} \pi_2$ and $\pi_1 \equiv_{\mathcal{K}}^{\psi_2} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K})$.

By the state-focus and conjunction-consistency properties, we can derive an equivalence on the existential quantification of a conjunction between a state and a path formula that allow to extract the first one from the scope of the quantifier. Similarly, we can extract a state formula from a universal quantification of a disjunction between this and a path formula. This property is simply an extension of what we have in the case of ungraded quantifications.

THEOREM 4.7 (LOCAL CONJUNCTION QUANTIFICATION). *Let $\equiv_{\cdot}^{\cdot}$ be a state-focused and conjunction-consistent equivalence relation. Moreover, let φ and ψ be a state and a path formula, respectively, and $g \in [1, \omega]$. Then, the following holds:* (i) $\mathsf{E}^{\geq g}(\varphi \wedge \psi) \equiv \varphi \wedge \mathsf{E}^{\geq g}\psi$ *and* (ii) $\mathsf{A}^{<g}(\varphi \vee \psi) \equiv \varphi \vee \mathsf{A}^{<g}\psi$.

PROOF. [Only if]. If $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\varphi \wedge \psi$ then $|(\mathrm{Pth}(\mathcal{K}, w_0, \varphi \wedge \psi)/{\equiv_{\mathcal{K}}^{\varphi\wedge\psi}})| \geq g$, where w_0 is the initial world of $\mathcal{K}$. The inequality implies $\mathrm{Pth}(\mathcal{K}, w_0, \varphi \wedge \psi) \neq \emptyset$, so, by Item iii of Proposition 3.1, there is a path $\pi \in \mathrm{Pth}(\mathcal{K}, w_0)$ such that $\mathcal{K}, \pi \models \varphi$ and, by Item ii of the same proposition, this means that $\mathcal{K}, w_0 \models \varphi$. Then, again by Item iii of Proposition 3.1, it is immediate to see that $\mathrm{Pth}(\mathcal{K}, w_0, \varphi \wedge \psi) = \mathrm{Pth}(\mathcal{K}, w_0, \psi)$. Moreover, by the state-focus property, we have that $\pi_1 \equiv_{\mathcal{K}}^{\varphi} \pi_2$, for all paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0)$. Now, by the conjunction-consistency property, we obtain that $\pi_1 \equiv_{\mathcal{K}}^{\varphi\wedge\psi} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$. At this point, $(\mathrm{Pth}(\mathcal{K}, w_0, \varphi \wedge \psi)/{\equiv_{\mathcal{K}}^{\varphi\wedge\psi}}) = (\mathrm{Pth}(\mathcal{K}, w_0, \psi)/{\equiv_{\mathcal{K}}^{\varphi\wedge\psi}}) = (\mathrm{Pth}(\mathcal{K}, w_0, \psi)/{\equiv_{\mathcal{K}}^{\psi}})$. Hence, $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\psi$ and consequently $\mathcal{K}, w_0 \models \varphi \wedge \mathsf{E}^{\geq g}\psi$.

[If]. If $\mathcal{K}, w_0 \models \varphi \wedge \mathsf{E}^{\geq g}\psi$, we have that $\mathcal{K}, w_0 \models \varphi$ and $|(\mathrm{Pth}(\mathcal{K}, w_0, \psi)/{\equiv_{\mathcal{K}}^{\psi}})| \geq g$. Then, by Items ii and iii of Proposition 3.1, it is immediate to see that $\mathrm{Pth}(\mathcal{K}, w_0, \psi) = \mathrm{Pth}(\mathcal{K}, w_0, \varphi \wedge \psi)$. Moreover, by the state-focus property, we have that $\pi_1 \equiv_{\mathcal{K}}^{\varphi} \pi_2$, for all paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0)$. Now, by the conjunction-consistency property, we obtain that $\pi_1 \equiv_{\mathcal{K}}^{\varphi\wedge\psi} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$. At this point, $(\mathrm{Pth}(\mathcal{K}, w_0, \psi)/{\equiv_{\mathcal{K}}^{\psi}}) = (\mathrm{Pth}(\mathcal{K}, w_0, \varphi \wedge \psi)/{\equiv_{\mathcal{K}}^{\psi}}) = (\mathrm{Pth}(\mathcal{K}, w_0, \varphi \wedge \psi)/{\equiv_{\mathcal{K}}^{\varphi\wedge\psi}})$. Hence, $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\varphi \wedge \psi$. □

It is interesting to note that, in order to prove the previous result, we do not need the full power of the conjunction consistency but a weaker property, which we denote *local conjunction consistency*, that only links the equivalence w.r.t. a conjunction of a state and a path formula to the equivalences w.r.t. the conjuncts. However, as we show later, we need the full power of the property when we have to reason about complex CTL* path formulas.

Consider again the two path formulas ψ_1 and ψ_2. We would like that, from a given world, the sum of the number of paths that satisfy ψ_1 together with that satisfying ψ_2 is not less than the number of paths that satisfy their disjunction. Suppose that there are only two paths that satisfy ψ_1 (resp., ψ_2) and are equivalent w.r.t. the same formula. Then, the two paths need to be equivalent w.r.t. $\psi_1 \vee \psi_2$, too. Hence, one way to ensure such a property is to ask that, whenever two paths are equivalent w.r.t. one formula, they are also equivalent w.r.t. its disjunctions. Moreover, we would like that both the number of paths that satisfy ψ_1 and ψ_2 are not greater than those satisfying $\psi_1 \vee \psi_2$. Hence, we need that paths satisfying ψ_1 (resp., ψ_2) and equivalent w.r.t. $\psi_1 \vee \psi_2$ are also equivalent w.r.t. ψ_1 (resp., ψ_2). So, we would like that two paths are equivalent w.r.t. a disjunction if and only if they are equivalent w.r.t. one of the two disjuncts.

Definition 4.9 (*Disjunction Consistency*). An equivalence relation $\equiv_{\mathcal{K}}^{\cdot}$ on paths is said to be *disjunction consistent* if and only if it holds that $\pi_1 \equiv_{\mathcal{K}}^{\psi_1\vee\psi_2} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi_1} \pi_2$ or $\pi_1 \equiv_{\mathcal{K}}^{\psi_2} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K})$.

In general, however, such a property contradicts the syntax-independence, state-focus, and the next- and weak next-consistency properties. Indeed, let $\psi_1 = \mathsf{X}\, p$ and $\psi_2 = \neg\, \mathsf{X}\, p$, for an atomic proposition $p \in \mathrm{AP}$. Then, $\psi_1 \vee \psi_2$ is equivalent to $\mathfrak{t}$. Consider now two paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w)$ such that $\mathcal{K}, (\pi_1)_1 \models p$ and $\mathcal{K}, (\pi_2)_1 \not\models p$, and so $(\pi_1)_1 \neq (\pi_2)_1$. Since the two paths have, in their second position, different successors of the origin, they are distinct w.r.t. ψ_1 and ψ_2 but they are identical w.r.t. $\psi_1 \vee \psi_2$, because of the state-focus and syntax-independence properties. In this example, the contradiction rises from the fact that the disjunction turns out to be a weaker property (a tautology) than the two base formulas. Hence, the formula is always satisfied and, since all choices over the paths are indifferent, they may be regarded as equivalent. Now, one may think that this is a problem related only to tautologies that rise from the disjunction. Unfortunately, this is not the case. Indeed, the disjunction may contain an hidden tautology that reveals itself only at some later points on the paths. For example, let $\psi_1 = \mathsf{X}\,\mathsf{X}\,p$ and $\psi_2 = \mathsf{X}\,\neg\,\mathsf{X}\,p$. Their disjunction is not a tautology, because it is not satisfied on paths of length 1. Consider now two paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w)$ such that $(\pi_1)_1 = (\pi_2)_1$, $\mathcal{K}, (\pi_1)_2 \models p$, and $\mathcal{K}, (\pi_2)_2 \not\models p$. The two paths are distinct w.r.t. ψ_1 and ψ_2 because they have distinct third nodes, but they are identical w.r.t. $\psi_1 \vee \psi_2 \equiv \mathsf{X}\,\mathfrak{t}$. It is easy to believe that the hidden tautology may be found arbitrary deeper in the formula, that is why the disjunction-consistency cannot hold in its entirety.

Since it is not possible to define in general an easy property that relates the equivalence on a disjunction to the equivalence on the component formulas, we restrict our observations to a case where the tautology derived from the disjunction can appear only at the first node of paths. Hence, we consider only disjunctions between a state φ and a path formula ψ. In such a case, two paths equivalent w.r.t. the disjunction $\varphi \vee \psi \equiv \varphi \vee \neg\varphi \equiv \mathfrak{t}$ are equivalent w.r.t. one of the two state formulas, too. In the next section, we actually prove that this property does not contradict the previous ones.

Definition 4.10 (*Local Disjunction Consistency*). An equivalence relation $\equiv^{\cdot}_{\mathcal{K}}$ on paths is said to be *local disjunction consistent* if and only if it holds that $\pi_1 \equiv^{\varphi\vee\psi}_{\mathcal{K}} \pi_2$ if and only if $\pi_1 \equiv^{\varphi}_{\mathcal{K}} \pi_2$ or $\pi_1 \equiv^{\psi}_{\mathcal{K}} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K})$, where φ is a state formula.

We further discuss an incidental property.

Consider a path formula ψ. Since in the semantics we only consider paths satisfying ψ when evaluating the truth nature of an existential or universal quantification, it is pointless to compare two paths if one of them does not satisfy ψ. However, suppose that there exist two paths π_1 and π_2 that do not satisfy a state formula φ, but that are equivalent w.r.t. φ. Also suppose that these paths satisfy a path formula ψ, but they are not equivalent w.r.t. ψ. Then, by local disjunction consistency the two paths would be equivalent w.r.t. $\varphi \vee \psi$, but it is unreasonable that there is only one path satisfying the disjunction while φ is not satisfied on them and there are two paths satisfying the formula ψ. In order to avoid such a problem, we may want to require that two paths are equivalent w.r.t. a formula only if they both satisfy it.

Definition 4.11 (*Satisfiability Constraint*). An equivalence relation $\equiv^{\cdot}_{\mathcal{K}}$ on paths is said to be *satisfiability constrained* if and only if it holds that if $\pi_1 \equiv^{\psi}_{\mathcal{K}} \pi_2$ then $\mathcal{K}, \pi_1 \models \psi$ and $\mathcal{K}, \pi_2 \models \psi$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K})$.

By the state-focus, local disjunction-consistency, and satisfiability-constraint properties, we can derive an equivalence on the quantification of a disjunction between a state and a path formula that allow to extract in a negated form the first one from the scope of the quantifier. Similarly, we can extract a negated state formula from a universal quantification of a conjunction between this and a path formula. Note that this property is not an extension of what we have in the case of ungraded quantifications.

THEOREM 4.8 (LOCAL DISJUNCTION QUANTIFICATION). *Let $\equiv^{\cdot}_{\cdot}$ be a state-focused, local disjunction-consistent, and satisfiability-constrained equivalence relation. Moreover, let φ and ψ be a state and a path formula, respectively, and $g \in [2, \omega]$. Then, the following holds:* (i) $\mathsf{E}^{\geq g}(\varphi \vee \psi) \equiv \neg\varphi \wedge \mathsf{E}^{\geq g}\psi$ *and* (ii) $\mathsf{A}^{<g}(\varphi \wedge \psi) \equiv \neg\varphi \vee \mathsf{A}^{<g}\psi$.

Proof. [Only if]. If $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\varphi \vee \psi$ then $|(\mathrm{Pth}(\mathcal{K}, w_0, \varphi \vee \psi)/\equiv_{\mathcal{K}}^{\varphi\vee\psi})| \geq g$, where w_0 is the initial world of $\mathcal{K}$. Suppose now by contradiction that $\mathcal{K}, w_0 \models \varphi$. Then, by the state-focus property, we have that $\pi_1 \equiv_{\mathcal{K}}^{\varphi} \pi_2$, for all paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0)$. So, by the local disjunction-consistency property, we obtain that $\pi_1 \equiv_{\mathcal{K}}^{\varphi\vee\psi} \pi_2$ and then that $|(\mathrm{Pth}(\mathcal{K}, w_0, \varphi \vee \psi)/\equiv_{\mathcal{K}}^{\varphi\vee\psi})| = 1 < g$, but this contradict the hypothesis. Hence, $\mathcal{K}, w_0 \not\models \varphi$, that is, $\mathcal{K}, w_0 \models \neg\varphi$. Then, by Item iv of Proposition 3.1, it is immediate to see that $\mathrm{Pth}(\mathcal{K}, w_0, \varphi \vee \psi) = \mathrm{Pth}(\mathcal{K}, w_0, \psi)$. Moreover, by the satisfiability-constraint property, we have that $\pi_1 \not\equiv_{\mathcal{K}}^{\varphi} \pi_2$, for all paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0)$. Now, again by the local disjunction-consistency property, we obtain that $\pi_1 \equiv_{\mathcal{K}}^{\varphi\vee\psi} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$. At this point, $(\mathrm{Pth}(\mathcal{K}, w_0, \varphi \vee \psi)/\equiv_{\mathcal{K}}^{\varphi\vee\psi}) = (\mathrm{Pth}(\mathcal{K}, w_0, \psi)/\equiv_{\mathcal{K}}^{\varphi\vee\psi}) = (\mathrm{Pth}(\mathcal{K}, w_0, \psi)/\equiv_{\mathcal{K}}^{\psi})$. Hence, $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\psi$ and consequently $\mathcal{K}, w_0 \models \neg\varphi \wedge \mathsf{E}^{\geq g}\psi$.

[If]. If $\mathcal{K}, w_0 \models \neg\varphi \wedge \mathsf{E}^{\geq g}\psi$, we have that $\mathcal{K}, w_0 \not\models \varphi$ and $|(\mathrm{Pth}(\mathcal{K}, w_0, \psi)/\equiv_{\mathcal{K}}^{\psi})| \geq g$. Then, by Item iv of Proposition 3.1, it is immediate to see that $\mathrm{Pth}(\mathcal{K}, w_0, \psi) = \mathrm{Pth}(\mathcal{K}, w_0, \varphi \vee \psi)$. Moreover, by the satisfiability-constraint property, we have that $\pi_1 \not\equiv_{\mathcal{K}}^{\varphi} \pi_2$, for all paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0)$. Now, by the local disjunction-consistency property, we obtain that $\pi_1 \equiv_{\mathcal{K}}^{\varphi\vee\psi} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$. At this point, $(\mathrm{Pth}(\mathcal{K}, w_0, \psi)/\equiv_{\mathcal{K}}^{\psi}) = (\mathrm{Pth}(\mathcal{K}, w_0, \varphi \vee \psi)/\equiv_{\mathcal{K}}^{\psi}) = (\mathrm{Pth}(\mathcal{K}, w_0, \varphi \vee \psi)/\equiv_{\mathcal{K}}^{\varphi\vee\psi})$. Hence, $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\varphi \vee \psi$. □

4.4. Main Properties

We now summarize all the previous properties in the single concept of adequacy.

Definition 4.12 (*Adequacy*). An equivalence relation $\equiv_{\mathcal{K}}^{\cdot}$ on paths is said to be *adequate* w.r.t. an equivalence structure $\cong^{\cdot}_{\cdot}$ if and only if it holds that it is (i) syntax independent, (ii) state focused, (iii) next consistent, (iv) weak next consistent w.r.t. $\cong^{\cdot}_{\cdot}$, (v) source dependent, (vi) conjunction consistent, (vii) local disjunction consistent, and (viii) satisfiability constrained.

Next theorem shows four exponential fixpoint expressions that extend to graded formulas the corresponding well-known results for ungraded ones. These interesting equivalences among GCTL formulas are useful to describe important properties of its semantics.

Theorem 4.9 (GCTL Fixpoint Equivalences). *Let $\equiv^{\cdot}_{\cdot}$ be an adequate equivalence relation. Moreover, let φ_1 and φ_2 be two state formulas and $g \in [2, \omega]$. Then, the following equivalences hold:*

(i) $\mathsf{E}^{\geq g}\varphi_1 \mathsf{U}\, \varphi_2 \equiv \neg\varphi_2 \wedge \varphi_1 \wedge (\mathsf{E\breve{X}}(g, \varphi_1 \mathsf{U}\, \varphi_2) \vee \mathsf{E}^{\geq 1}\mathsf{X}\, \mathsf{E}^{\geq g}\varphi_1 \mathsf{U}\, \varphi_2)$;
(ii) $\mathsf{E}^{\geq g}\varphi_1 \mathsf{R}\, \varphi_2 \equiv \varphi_2 \wedge \neg\varphi_1 \wedge (\mathsf{E\breve{X}}(g, \varphi_1 \mathsf{R}\, \varphi_2) \vee \mathsf{E}^{\geq 1}\mathsf{X}\, \mathsf{E}^{\geq g}\varphi_1 \mathsf{R}\, \varphi_2)$;
(iii) $\mathsf{A}^{<g}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2 \equiv \varphi_2 \vee \neg\varphi_1 \vee \mathsf{A}\tilde{\breve{\mathsf{X}}}(g, \varphi_1 \tilde{\mathsf{U}}\, \varphi_2) \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\, \mathsf{A}^{<g}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2$;
(iv) $\mathsf{A}^{<g}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2 \equiv \neg\varphi_2 \vee \varphi_1 \vee \mathsf{A}\tilde{\breve{\mathsf{X}}}(g, \varphi_1 \tilde{\mathsf{R}}\, \varphi_2) \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\, \mathsf{A}^{<g}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2$.

Proof. To show Item i (respectively, ii), it is possible to apply to the formula $\mathsf{E}^{\geq g}\varphi_1 \mathsf{U}\, \varphi_2$ (respectively, $\mathsf{E}^{\geq g}\varphi_1 \mathsf{R}\, \varphi_2$) the following chain of equivalences: Item i (respectively, ii) of Corollary 4.1 and Theorems 4.8 (respectively, 4.7), 4.7 (respectively, 4.8), 4.5, and 4.6. At the same way, to show Item iii (respectively, iv), it is possible to apply to the formula $\mathsf{A}^{<g}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2$ (respectively, $\mathsf{A}^{<g}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2$) the following sequence of equivalences: Item vii (respectively, viii) of Corollary 4.1, and Theorems 4.7 (respectively, 4.8), 4.8 (respectively, 4.7), 4.5, and 4.6. □

In the following, we use the four macros $\mathsf{EU}(g, \varphi_1, \varphi_2, Y)$, $\mathsf{ER}(g, \varphi_1, \varphi_2, Y)$, $\mathsf{A\tilde{U}}(g, \varphi_1, \varphi_2, Y)$, and $\mathsf{A\tilde{R}}(g, \varphi_1, \varphi_2, Y)$ defined below, to represent in short the expansion formulas for the existential U and R and the universal $\tilde{\mathsf{U}}$ and $\tilde{\mathsf{R}}$ temporal operators derived in the previous theorem and in Items i, ii, iii, and iv of Proposition 3.3.

—$\mathsf{EU}(g, \varphi_1, \varphi_2, Y) \triangleq \begin{cases} \varphi_2 \vee \varphi_1 \wedge \mathsf{E}^{\geq 1}\mathsf{X}\, Y, & \text{if } g = 1; \\ \neg\varphi_2 \wedge \varphi_1 \wedge (\mathsf{E}\mathcal{X}(g, \varphi_1 \mathsf{U}\, \varphi_2) \vee \mathsf{E}^{\geq 1}\mathsf{X}\, Y), & \text{otherwise.} \end{cases}$

—$\mathsf{ER}(g, \varphi_1, \varphi_2, Y) \triangleq \begin{cases} \varphi_2 \wedge (\varphi_1 \vee \mathsf{E}^{\geq 1}\mathsf{X}\, Y), & \text{if } g = 1; \\ \varphi_2 \wedge \neg\varphi_1 \wedge (\mathsf{E}\mathcal{X}(g, \varphi_1 \mathsf{R}\, \varphi_2) \vee \mathsf{E}^{\geq 1}\mathsf{X}\, Y), & \text{otherwise.} \end{cases}$

—$\mathsf{A}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, Y) \triangleq \begin{cases} \varphi_2 \vee \varphi_1 \wedge \mathsf{A}^{<1}\mathsf{X}\, Y, & \text{if } g = 1; \\ \varphi_2 \vee \neg\varphi_1 \vee \mathsf{A}\tilde{\mathcal{X}}(g, \varphi_1 \tilde{\mathsf{U}}\, \varphi_2) \wedge \mathsf{A}^{<1}\mathsf{X}\, Y, & \text{otherwise.} \end{cases}$

—$\mathsf{A}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, Y) \triangleq \begin{cases} \varphi_2 \wedge (\varphi_1 \vee \mathsf{A}^{<1}\mathsf{X}\, Y), & \text{if } g = 1; \\ \neg\varphi_2 \vee \varphi_1 \vee \mathsf{A}\tilde{\mathcal{X}}(g, \varphi_1 \tilde{\mathsf{R}}\, \varphi_2) \wedge \mathsf{A}^{<1}\mathsf{X}\, Y, & \text{otherwise.} \end{cases}$

It is immediate to see that $|\mathsf{EU}(g, \varphi_1, \varphi_2, Y)| = |\mathsf{ER}(g, \varphi_1, \varphi_2, Y)| = \Theta(|Y| + (|\varphi_1| + |\varphi_2| + 5) \cdot 2^{k \cdot \sqrt{g}})$ and $|\mathsf{A}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, Y)| = |\mathsf{A}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, Y)| = \Theta(|Y| + (|\varphi_1| + |\varphi_2| + 5) \cdot 2^{k \cdot \sqrt{g-1}})$, for a constant k. Moreover, for all $g \in [1, \omega]$, it holds that

—$\mathsf{E}^{\geq g}\varphi_1 \mathsf{U}\, \varphi_2 \equiv \mathsf{EU}(g, \varphi_1, \varphi_2, \mathsf{E}^{\geq g}\varphi_1 \mathsf{U}\, \varphi_2)$,
—$\mathsf{E}^{\geq g}\varphi_1 \mathsf{R}\, \varphi_2 \equiv \mathsf{ER}(g, \varphi_1, \varphi_2, \mathsf{E}^{\geq g}\varphi_1 \mathsf{R}\, \varphi_2)$,
—$\mathsf{A}^{<g}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2 \equiv \mathsf{A}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, \mathsf{A}^{<g}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2)$,
—$\mathsf{A}^{<g}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2 \equiv \mathsf{A}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, \mathsf{A}^{<g}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2)$.

Differently from the previous cases, we cannot hope to obtain similar general fixpoint equivalences for the existential $\tilde{\mathsf{U}}$ and $\tilde{\mathsf{R}}$ and the universal U and R temporal operators. This is due to the fact that we do not have general equivalences between the quantifications of $\mathsf{X}\, \psi$ and those of $\tilde{\mathsf{X}}\, \psi$. The next theorem shows the four exponential fixpoint properties we are able to derive for these cases.

THEOREM 4.10 (GCTL ALMOST FIXPOINT EQUIVALENCES). *Let $\equiv_{\cdot}$ be an adequate equivalence relation w.r.t. the equivalence structure $\cong_{\cdot}^{\cdot}$. Moreover, let $\mathcal{K}$ be a* KS*, w_0 its initial world, φ_1 and φ_2 be two state formulas, and $g \in [2, \omega]$. Then, the following hold:*

(i) $\mathcal{K} \models \mathsf{E}^{\geq g}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2$ *if and only if* $\mathcal{K} \models \neg\varphi_2 \wedge \varphi_1 \wedge (\mathsf{E}\mathcal{X}(g, \varphi_1 \tilde{\mathsf{U}}\, \varphi_2) \vee \mathsf{E}^{\geq 1}\mathsf{X}\, \mathsf{E}^{\geq g}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2)$ *and* $\tilde{\mathsf{X}}\, \varphi_1 \tilde{\mathsf{U}}\, \varphi_2$ *is not an* $\cong_{\mathcal{K}}^{w_0}$*-tautology;*
(ii) $\mathcal{K} \models \mathsf{E}^{\geq g}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2$ *if and only if* $\mathcal{K} \models \varphi_2 \wedge \neg\varphi_1 \wedge (\mathsf{E}\mathcal{X}(g, \varphi_1 \tilde{\mathsf{R}}\, \varphi_2) \vee \mathsf{E}^{\geq 1}\mathsf{X}\, \mathsf{E}^{\geq g}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2)$ *and* $\tilde{\mathsf{X}}\, \varphi_1 \tilde{\mathsf{R}}\, \varphi_2$ *is not an* $\cong_{\mathcal{K}}^{w_0}$*-tautology;*
(iii) $\mathcal{K} \models \mathsf{A}^{<g}\varphi_1 \mathsf{U}\, \varphi_2$ *if and only if* $\mathcal{K} \models \varphi_2 \vee \neg\varphi_1 \vee \mathsf{A}\tilde{\mathcal{X}}(g, \varphi_1 \mathsf{U}\, \varphi_2) \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\, \mathsf{A}^{<g}\varphi_1 \mathsf{U}\, \varphi_2$ *or* $\neg\mathsf{X}\, \varphi_1 \mathsf{U}\, \varphi_2$ *is an* $\cong_{\mathcal{K}}^{w_0}$*-tautology;*
(iv) $\mathcal{K} \models \mathsf{A}^{<g}\varphi_1 \mathsf{R}\, \varphi_2$ *if and only if* $\mathcal{K} \models \neg\varphi_2 \vee \varphi_1 \vee \mathsf{A}\tilde{\mathcal{X}}(g, \varphi_1 \mathsf{R}\, \varphi_2) \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\, \mathsf{A}^{<g}\varphi_1 \mathsf{R}\, \varphi_2$ *or* $\neg\mathsf{X}\, \varphi_1 \mathsf{R}\, \varphi_2$ *is an* $\cong_{\mathcal{K}}^{w_0}$*-tautology.*

PROOF. To show Item (i) (respectively, (ii), it is possible to apply to the formula $\mathsf{E}^{\geq g}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2$ (respectively, $\mathsf{E}^{\geq g}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2$) the following chain of equivalences: Item iii (respectively, iv) of Corollary 4.1, and Theorems 4.8 (respectively, 4.7), 4.7 (respectively, 4.8), 4.3, 4.5, and 4.6. At the same way, to show Item (iii) (respectively, (iv), it is possible to apply to the formula $\mathsf{A}^{<g}\varphi_1 \mathsf{U}\, \varphi_2$ (respectively, $\mathsf{A}^{<g}\varphi_1 \mathsf{R}\, \varphi_2$) the following sequence of equivalences: Item (v) (respectively, (vi)) of Corollary 4.1, and Theorems 4.7 (respectively, 4.8), 4.8 (respectively, 4.7), 4.3, 4.5, and 4.6. □

As for the previous cases, in the following, we use the macros $\mathsf{E}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, Y, \varphi)$, $\mathsf{E}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, Y, \varphi)$, $\mathsf{AU}(g, \varphi_1, \varphi_2, Y, \varphi)$, and $\mathsf{AR}(g, \varphi_1, \varphi_2, Y, \varphi)$ defined in the following, to represent in short the expansion formulas for the existential $\tilde{\mathsf{U}}$ and $\tilde{\mathsf{R}}$ and the universal U and R temporal operators derived in the previous theorem and in Items iii, iv, v, and vi of Proposition 3.3.

—$\mathsf{E}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, Y, \varphi) \triangleq \begin{cases} \varphi_2 \vee \varphi_1 \wedge (\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}^{\geq 1}\mathsf{X}\,Y), & \text{if } g = 1; \\ \neg\varphi_2 \wedge \varphi_1 \wedge (\mathsf{E}\tilde{\mathsf{X}}(g, \varphi_1 \tilde{\mathsf{U}}\, \varphi_2) \vee \mathsf{E}^{\geq 1}\mathsf{X}\,Y) \wedge \varphi, & \text{otherwise.} \end{cases}$

—$\mathsf{E}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, Y, \varphi) \triangleq \begin{cases} \varphi_2 \wedge (\varphi_1 \vee \mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}^{\geq 1}\mathsf{X}\,Y), & \text{if } g = 1; \\ \varphi_2 \wedge \neg\varphi_1 \wedge (\mathsf{E}\tilde{\mathsf{X}}(g, \varphi_1 \tilde{\mathsf{R}}\, \varphi_2) \vee \mathsf{E}^{\geq 1}\mathsf{X}\,Y) \wedge \varphi, & \text{otherwise.} \end{cases}$

—$\mathsf{AU}(g, \varphi_1, \varphi_2, Y, \varphi) \triangleq \begin{cases} \varphi_2 \vee \varphi_1 \wedge \mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t} \wedge \mathsf{A}^{<1}\mathsf{X}\,Y, & \text{if } g = 1; \\ \varphi_2 \vee \neg\varphi_1 \vee \mathsf{A}\tilde{\mathsf{X}}(g, \varphi_1 \mathsf{U}\, \varphi_2) \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\,Y \vee \varphi, & \text{otherwise.} \end{cases}$

—$\mathsf{AR}(g, \varphi_1, \varphi_2, Y, \varphi) \triangleq \begin{cases} \varphi_2 \wedge (\varphi_1 \vee \mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t} \wedge \mathsf{A}^{<1}\mathsf{X}\,Y), & \text{if } g = 1; \\ \neg\varphi_2 \vee \varphi_1 \vee \mathsf{A}\tilde{\mathsf{X}}(g, \varphi_1 \mathsf{R}\, \varphi_2) \wedge \mathsf{A}^{<1}\tilde{\mathsf{X}}\,Y \vee \varphi, & \text{otherwise.} \end{cases}$

It is immediate to see that $|\mathsf{E}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, Y, \varphi)| = |\mathsf{E}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, Y, \varphi)| = \Theta(|Y| + |\varphi| + (|\varphi_1| + |\varphi_2| + 5) \cdot 2^{k \cdot \sqrt{g}})$ and $|\mathsf{AU}(g, \varphi_1, \varphi_2, Y, \varphi)| = |\mathsf{AR}(g, \varphi_1, \varphi_2, Y, \varphi)| = \Theta(|Y| + |\varphi| + (|\varphi_1| + |\varphi_2| + 5) \cdot 2^{k \cdot \sqrt{g-1}})$, for a constant k. As yet noted before, there are no general equivalences that directly link the formulas $\mathsf{E}^{\geq g}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2$, $\mathsf{E}^{\geq g}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2$, $\mathsf{A}^{<g}\varphi_1 \mathsf{U}\, \varphi_2$, and $\mathsf{A}^{<g}\varphi_1 \mathsf{R}\, \varphi_2$ with their expansions $\mathsf{E}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, \mathsf{E}^{\geq g}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2, \varphi)$, $\mathsf{E}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, \mathsf{E}^{\geq g}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2, \varphi)$, $\mathsf{AU}(g, \varphi_1, \varphi_2, \mathsf{A}^{<g}\varphi_1 \mathsf{U}\, \varphi_2, \varphi)$, and $\mathsf{AR}(g, \varphi_1, \varphi_2, \mathsf{A}^{<g}\varphi_1 \mathsf{R}\, \varphi_2, \varphi)$. Note that here the metavariable φ can be used at the same way of that of the macro $\mathsf{E}\tilde{\mathsf{X}}(g, \psi, \varphi)$.

Finally, we show a fundamental equivalence that allows us to extract all state formulas from the scope of a quantification of a generic GCTL* path formula.

THEOREM 4.11 (GCTL* PATH EXPANSION EQUIVALENCE). *Let $\equiv_{\cdot}^{\cdot}$ be a syntax-independent, state-focused, conjunction-consistent, local disjunction-consistent, and satisfiability-constrained equivalence relation. Moreover, let φ_i and ψ_i be, respectively, k state and path formulas, $\mathsf{Op}_i \in \{\mathsf{X}, \tilde{\mathsf{X}}\}$, and $g \in [1, \omega]$. Then, the following equivalences hold, where $\psi = \bigwedge_{i=1}^{k}(\varphi_i \vee \mathsf{Op}_i \psi_i)$, $\varphi_I = \bigwedge_{i \in I} \varphi_i \wedge \bigwedge_{i \in [1,k] \setminus I} \neg\varphi_i$ and $\psi_I = \mathsf{Op} \bigwedge_{i \in [1,k] \setminus I} \psi_i$ with $\mathsf{Op} \in \{\mathsf{X}, \tilde{\mathsf{X}}\}$ and $\mathsf{Op} = \mathsf{X}$ if and only if there is $i \in [1, k] \setminus I$ such that $\mathsf{Op}_i = \mathsf{X}$.*

(1) $\mathsf{E}^{\geq g}\psi \equiv \bigvee_{I \subseteq [1,k]} \varphi_I \wedge \mathsf{E}^{\geq g}\psi_I$;
(2) $\mathsf{A}^{<g}\neg\psi \equiv \bigvee_{I \subseteq [1,k]} \varphi_I \wedge \mathsf{A}^{<g}\neg\psi_I$.

PROOF. We have to prove that $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\psi$ if and only if $\mathcal{K}, w_0 \models \bigvee_{I \subseteq [1,k]} \varphi_I \wedge \mathsf{E}^{\geq g}\psi_I$ (resp., $\mathcal{K}, w_0 \models \mathsf{A}^{<g}\neg\psi$ if and only if $\mathcal{K}, w_0 \models \bigvee_{I \subseteq [1,k]} \varphi_I \wedge \mathsf{A}^{<g}\neg\psi_I$), where w_0 is the initial world of $\mathcal{K}$, for all Ks $\mathcal{K}$. First, let $I \subseteq [1, k]$ be the set of indexes of just the state formulas φ_i that are true on $\mathcal{K}$, that is, such that (i) $\mathcal{K}, w_0 \models \varphi_i$, for all $i \in I$, and (ii) $\mathcal{K}, w_0 \not\models \varphi_i$, for all $i \in [1, k] \setminus I$. Thus, $\mathcal{K}, w_0 \models \varphi_I$. Note that such a set is uniquely determined by the Ks $\mathcal{K}$.

By Items iii and iv of Proposition 3.1, it holds that $\mathrm{Pth}(\mathcal{K}, w_0, \psi) = \mathrm{Pth}(\mathcal{K}, w_0, \psi_I)$. What remains to prove is that $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi_I} \pi_2$, for all $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K}, w_0)$. By the conjunction-consistency property, we have that $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$ if and only if, for all $i \in [1, k]$, it holds that $\pi_1 \equiv_{\mathcal{K}}^{\varphi_i \vee \mathsf{Op}_i \psi_i} \pi_2$. Thus, by the local disjunction-consistency property, we obtain that $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$ if and only if, for all $i \in [1, k]$, it holds that $\pi_1 \equiv_{\mathcal{K}}^{\varphi_i} \pi_2$ or $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{Op}_i \psi_i} \pi_2$. Now, if $i \in I$, by the state-focus property, it holds that $\pi_1 \equiv_{\mathcal{K}}^{\varphi_i} \pi_2$. On the contrary, if $i \in [1, k] \setminus I$, by the satisfiability-constraint property, it holds that $\pi_1 \not\equiv_{\mathcal{K}}^{\varphi_i} \pi_2$. Hence, the previous coimplication between $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$ and its expansion can be simplified as follows: $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$ if and only if, for all $i \in [1, k] \setminus I$, it holds that $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{Op}_i \psi_i} \pi_2$. At this point, again by the conjunction-consistency property, we have that $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\bigwedge_{i \in [1,k] \setminus I} \mathsf{Op}_i \psi_i} \pi_2$. Now, it is easy to note that $\bigwedge_{i \in [1,k] \setminus I} \mathsf{Op}_i \psi_i \equiv \psi_I$. So, by the syntax-independence property, we can further simplify the previous coimplication in $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi_I} \pi_2$, obtaining directly that $(\mathrm{Pth}(\mathcal{K}, w_0, \psi)/\equiv_{\mathcal{K}}^{\psi}) = (\mathrm{Pth}(\mathcal{K}, w_0, \psi_I)/\equiv_{\mathcal{K}}^{\psi_I})$. Thus, the assumption $\mathcal{K}, w_0 \models \varphi_I$ implies that $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\psi$ if and only if $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\psi_I$ (respectively, $\mathcal{K}, w_0 \models \mathsf{A}^{<g}\neg\psi$ if and only if $\mathcal{K}, w_0 \models \mathsf{A}^{<g}\neg\psi_I$).

Now, on one hand, it is easy to see that, for each Ks $\mathcal{K}$, there is a set $I \subseteq [1, k]$ such that $\mathcal{K}, w_0 \models \varphi_I$ and so $\mathsf{E}^{\geq g}\psi \Rightarrow \bigvee_{I\subseteq[1,k]} \varphi_I \wedge \mathsf{E}^{\geq g}\psi_I$ (resp., $\mathsf{A}^{<g}\neg\psi \Rightarrow \bigvee_{I\subseteq[1,k]} \varphi_I \wedge \mathsf{A}^{<g}\neg\psi_I$). On the other hand, the existence of a set $I \subseteq [1, k]$ such that $\mathcal{K}, w_0 \models \varphi_I$ and $\mathcal{K}, w_0 \models \mathsf{E}^{\geq g}\psi_I$ (resp., $\mathcal{K}, w_0 \models \mathsf{A}^{<g}\neg\psi_I$) implies $\mathsf{E}^{\geq g}\psi$ (respectively, $\mathsf{A}^{<g}\neg\psi$), that is, $\bigvee_{I\subseteq[1,k]} \varphi_I \wedge \mathsf{E}^{\geq g}\psi_I \Rightarrow \mathsf{E}^{\geq g}\psi$ (resp., $\bigvee_{I\subseteq[1,k]} \varphi_I \wedge \mathsf{A}^{<g}\neg\psi_I \Rightarrow \mathsf{A}^{<g}\neg\psi$). Hence, the thesis follows. □

It may be interesting to observe that the previous result is a generalization of Theorems 4.7 and 4.8 that can be obtained as the limit cases in which there are no conjunctions or disjunctions, respectively. Moreover, it is important to note that, differently from the case of the local conjunction quantification, here we need the full power of the conjunction-consistency property in order to prove this equivalence.

5. PREFIX PATH EQUIVALENCE

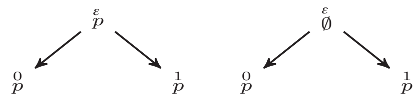
In this section, we introduce a suitable path equivalence relation that satisfies all the previously discussed properties. Hence, we show that those properties are not contradictory, by presenting one of the possible meaningful graded computation tree logics. In the sequel of the paper, we only refer to GCTL* under this specific equivalence relation.

5.1. Definition and Properties

In the definition of the GCTL* semantics, we use a generic equivalence relation $\equiv_{\cdot}^{\cdot}$ on paths that allows us to count how many ways a structure has to satisfy a path formula. So, two paths should be considered equivalent when they represent only one way to perform according to that formula. For many formulas, such a way results to be their common finite prefix. For example, all paths that satisfy $\mathsf{X}\, p$ and have the first two nodes in common may be regarded as equivalent because the first two nodes constitute the one sought way to satisfy the formula. For some other formula like $\tilde{\mathsf{X}}\, p$, the ways to satisfy it are less clear. For example, consider two paths π_1 and π_2 with only the starting node in common, such that the first satisfies $\mathsf{X}\, p$ while the latter $\mathsf{X}\, \neg p$. Then, the common node, if taken alone, that is, without its successors, may be considered as a path satisfying $\tilde{\mathsf{X}}\, p$. So, the two paths would be equivalent. However, this looks unreasonable because π_2 does not satisfy $\tilde{\mathsf{X}}\, p$ and, thus, the common prefix failed to ensure the conservativeness of the satisfiability for this formula. Hence, a common prefix between two paths may be considered as a way to satisfy a path formula, if it satisfies the formula and somehow it allows us to deduce that this formula is true on all paths with that prefix in the structure. The following definition of the equivalence relation among paths formally captures the previous idea.

Definition 5.1 (*Prefix Equivalence*). Two paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K})$ are *prefix equivalent* w.r.t. a path formula ψ, in symbols $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$, if and only if either $\pi_1 = \pi_2$ or (i) the common track $\rho = \mathsf{pfx}(\pi_1, \pi_2)$ of π_1 and π_2 is not empty and (ii) $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \psi$, for every path/track $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$.

Observe that when two paths are distinct w.r.t. $\equiv_{\mathcal{K}}^{\psi}$, there are always at least two successors of the last node of their common prefix. Hence, the Ks $\mathcal{K}$ is never allowed to stop its computations at that node, that is, the common prefix is a track but not a path in $\mathcal{K}$.



(a) $\mathcal{T}$. (b) $\mathcal{T}'$.

Fig. 2. Two finite KTS.

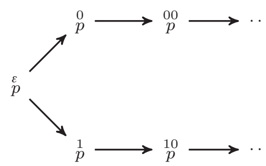


Fig. 3. An infinite KT.

We now give few simple examples of the behavior of GCTL* under the use of the prefix equivalence.

Consider a finite KT $\mathcal{T}$ having just three nodes all labeled by p, the root and its two successors (see Figure 2). Also, consider the formula $\varphi = \mathsf{E}^{\geq 2}\mathsf{F}\ p$. Because of the definition of the equivalence, the only two paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{T}, \varepsilon)$ of length two satisfying $\mathsf{F}\ p$ are equivalent, since the common prefix $\rho = \mathsf{pfx}(\pi_1, \pi_2)$ containing just the root satisfies the formula too. Hence, $\mathcal{T} \not\models \varphi$. On the contrary, if we take a tree $\mathcal{T}'$ that is the same of $\mathcal{T}$, but with its root not labeled with p, we obtain that $\mathcal{T}' \models \varphi$, since $\mathcal{T}', \rho \not\models \mathsf{F}\ p$. This means that the particular equivalence allows us to count as different events only their first appearance along the paths.

Consider now the formula $\varphi = \mathsf{E}^{\geq 2}\mathsf{G}\ p$ and an infinite KT $\mathcal{T}$ having just two paths all labeled by p (see Figure 3). Since $\mathsf{G}\ p$ cannot be satisfied on a track or finite path, we have that $\mathcal{T}, \rho \not\models \mathsf{G}\ p$, so the two infinite paths are not equivalent w.r.t. this formula, which implies that $\mathcal{T} \models \varphi'$. On the contrary, if we take $\varphi' = \mathsf{E}^{\geq 2}\tilde{\mathsf{G}}\ p$, then we obtain $\mathcal{T} \not\models \varphi'$, since each track and path completely labeled with p satisfies $\tilde{\mathsf{G}}\ p$.

We now define a new equivalence between path formulas that results to be compatible with the chosen prefix equivalence. Its definition, in particular, takes into account a KS $\mathcal{K}$ and one of its worlds w in which we want to verify that the two formulas under exam are interchangeable for the logic.

Definition 5.2 (*Structure Formula Equivalence*). Let $\mathcal{K}$ be a KS, w one of its worlds, and ψ_1 and ψ_2 be two path formulas. Then, ψ_1 is *structurally equivalent* to ψ_2 w.r.t. $\mathcal{K}$ and w, in symbols $\psi_1 \cong^{w}_{\mathcal{K}} \psi_2$, if and only if, for all paths/tracks $\pi \in (\mathrm{Pth}(\mathcal{K}, w) \cup \mathrm{Trk}(\mathcal{K}, w))$, it holds that $\mathcal{K}, \pi \models \psi_1$ if and only if $\mathcal{K}, \pi \models \psi_2$.

Observe that $\cong^{\cdot}_{\cdot}$ is an equivalence structure according to definition 4.4.

The following theorem shows that the prefix path relation satisfies the adequacy property defined in the previous section, if we consider the structure formula equivalence when we have to deal with the weak next operator.

THEOREM 5.1 (PREFIX EQUIVALENCE ADEQUACY). *The prefix equivalence relation is* adequate *w.r.t.* $\cong^{\cdot}_{\cdot}$.

PROOF. All the equivalence properties we want to show express that a given property on two paths implies a derived property on the same paths. So they are trivially satisfied when they concern two identical paths. For this reason in the following, we make the assumption that the two paths $\pi_1, \pi_2 \in \mathrm{Pth}(\mathcal{K})$ involved in the proof are distinct. Moreover, we use $\rho = \mathsf{pfx}(\pi_1, \pi_2)$ to indicate their common prefix.

(i) (Syntax independence). For $i \in \{1, 2\}$, if $\pi_1 \equiv_{\mathcal{K}}^{\psi_i} \pi_2$, then (i) $\rho \neq \varepsilon$ and (ii) $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \psi_i$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Since $\psi_1 \equiv \psi_2$, by Item (i) of Proposition 3.1, we obtain then that $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \psi_{3-i}$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Hence, $\pi_1 \equiv_{\mathcal{K}}^{\psi_{3-i}} \pi_2$.

(ii) (State focus). Assume that $(\pi_1)_0 = (\pi_2)_0$, thus obtaining $\rho \neq \varepsilon$. Since φ is a state formula, by Item (ii) of Proposition 3.1, we have that $\mathcal{K}, (\rho)_0 \models \varphi$ implies $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \varphi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Hence, $\pi_1 \equiv_{\mathcal{K}}^{\varphi} \pi_2$.

(iii) (Next consistency). Assume that $(\pi_1)_0 = (\pi_2)_0$. Then, it is immediate to see that $\rho \neq \varepsilon$ and $\rho_{\geq 1} = \mathsf{pfx}((\pi_1)_{\geq 1}, (\pi_2)_{\geq 1})$ is the common prefix of the suffixes of the two paths π_1 and π_2. [Only if]. If $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{X}\,\psi} \pi_2$, then $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \mathsf{X}\,\psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Since $\mathsf{lst}(\rho) \in \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho))$, we have that $\mathcal{K}, \rho \cdot \varepsilon \models \mathsf{X}\,\psi$, that is, $\mathcal{K}, \rho \models \mathsf{X}\,\psi$ and so, $\rho_{\geq 1} \neq \varepsilon$, by Item v of Proposition 3.1. Moreover, by the same item, one can note that $\mathcal{K}, (\rho \cdot \pi_{\geq 1})_{\geq 1} \models \psi$, that is, $\mathcal{K}, \rho_{\geq 1} \cdot \pi_{\geq 1} \models \psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho))) = (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1})) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1})))$. Hence, $(\pi_1)_{\geq 1} \equiv_{\mathcal{K}}^{\psi} (\pi_2)_{\geq 1}$. [If]. If $(\pi_1)_{\geq 1} \equiv_{\mathcal{K}}^{\psi} (\pi_2)_{\geq 1}$, then $\mathcal{K}, \rho_{\geq 1} \cdot \pi_{\geq 1} \models \psi$, that is, $\mathcal{K}, (\rho \cdot \pi_{\geq 1})_{\geq 1} \models \psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1})) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1})))$. Now, by Item v of Proposition 3.1, one can note that $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \mathsf{X}\,\psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1})) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1}))) = (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Hence, $\pi_1 \equiv_{\mathcal{K}}^{\mathsf{X}\,\psi} \pi_2$.

(iv) (Weak next consistency). Assume that $(\pi_1)_0 = (\pi_2)_0$. As in the previous item, we have that $\rho \neq \varepsilon$ and $\rho_{\geq 1} = \mathsf{pfx}((\pi_1)_{\geq 1}, (\pi_2)_{\geq 1})$. [Only if]. If $\pi_1 \equiv_{\mathcal{K}}^{\tilde{\mathsf{X}}\,\psi} \pi_2$, then $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \tilde{\mathsf{X}}\,\psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Now, suppose that $\tilde{\mathsf{X}}\,\psi$ is not an $\simeq_{\mathcal{K}}^{(\rho)_0}$-tautology. Then, it is possible to see that $\rho_{\geq 1} \neq \varepsilon$. Indeed, suppose by contradiction that $\rho_{\geq 1} = \varepsilon$ and let $\pi \in (\mathrm{Pth}(\mathcal{K}, (\rho)_0) \cup \mathrm{Trk}(\mathcal{K}, (\rho_0))$ be the path/track not satisfying $\tilde{\mathsf{X}}\,\psi$, that is, such that $\mathcal{K}, \pi \not\models \tilde{\mathsf{X}}\,\psi$. Since $(\rho)_0 = \mathsf{lst}(\rho)$, it is immediate to see that $\pi = \rho \cdot \pi_{\geq 1}$, so we have that $\mathcal{K}, \rho \cdot \pi_{\geq 1} \not\models \tilde{\mathsf{X}}\,\psi$, and this is in contradiction with the equivalence $\pi_1 \equiv_{\mathcal{K}}^{\tilde{\mathsf{X}}\,\psi} \pi_2$. At this point, by Item (vi) of Proposition 3.1, one can note that $\mathcal{K}, \rho_{\geq 1} \cdot \pi_{\geq 1} \models \psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho))) = (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1})) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1})))$. Hence, $(\pi_1)_{\geq 1} \equiv_{\mathcal{K}}^{\psi} (\pi_2)_{\geq 1}$. [If]. On one hand, if $\tilde{\mathsf{X}}\,\psi$ is an $\simeq_{\mathcal{K}}^{(\rho)_0}$-tautology, then all paths/tracks $\pi \in (\mathrm{Pth}(\mathcal{K}, (\rho)_0) \cup \mathrm{Trk}(\mathcal{K}, (\rho)_0))$ satisfy $\tilde{\mathsf{X}}\,\psi$, that is, $\mathcal{K}, \pi \models \tilde{\mathsf{X}}\,\psi$. Thus, $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \tilde{\mathsf{X}}\,\psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Hence, $\pi_1 \equiv_{\mathcal{K}}^{\tilde{\mathsf{X}}\,\psi} \pi_2$. On the other hand, if $(\pi_1)_{\geq 1} \equiv_{\mathcal{K}}^{\psi} (\pi_2)_{\geq 1}$, then $\mathcal{K}, \rho_{\geq 1} \cdot \pi_{\geq 1} \models \psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1})) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1})))$. Now, by Item vi of Proposition 3.1, one can note that $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \tilde{\mathsf{X}}\,\psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1})) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho_{\geq 1}))) = (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Hence, $\pi_1 \equiv_{\mathcal{K}}^{\tilde{\mathsf{X}}\,\psi} \pi_2$.

(v) (Source dependence). By definition, if the two paths π_1 and π_2 have no starting node in common, that is, $(\pi_1)_0 \neq (\pi_2)_0$, they cannot be prefix equivalent because $\rho = \varepsilon$, that is, they do not have any nonempty prefix in common at all.

(vi) (Conjunction consistency). Let $\psi = \psi_1 \wedge \psi_2$. Then, it holds that $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$ if and only if (i) $\rho \neq \varepsilon$ and (ii) $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. By Item (iii) of Proposition 3.1, the condition *(ii)* is equivalent to $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \psi_i$, for all $i \in \{1, 2\}$. Hence, $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$ if and only if $\pi_1 \equiv_{\mathcal{K}}^{\psi_1} \pi_2$ and $\pi_1 \equiv_{\mathcal{K}}^{\psi_2} \pi_2$.

(vii) (Local disjunction consistency). Let $\psi = \varphi \vee \psi'$, where φ is a state formula. [Only if]. If $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$, then *(i)* $\rho \neq \varepsilon$ and *(ii)* $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. First suppose that $\mathcal{K}, (\rho)_0 \models \varphi$. Then, by the state-focus property, we obtain that $\pi_1 \equiv_{\mathcal{K}}^{\varphi} \pi_2$. Suppose now that $\mathcal{K}, (\rho)_0 \not\models \varphi$. By Item (ii) of Proposition 3.1, we have that $\mathcal{K}, \rho \cdot \pi_{\geq 1} \not\models \varphi$, for all $\pi \in (\text{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \text{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$, and so, by Item (iv) of Proposition 3.1, we obtain that $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \psi'$, for all $\pi \in (\text{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \text{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Consequently, we obtain that $\pi_1 \equiv_{\mathcal{K}}^{\psi'} \pi_2$. [If]. If $\pi_1 \equiv_{\mathcal{K}}^{\varphi} \pi_2$ (respectively, $\pi_1 \equiv_{\mathcal{K}}^{\psi'} \pi_2$), then (i) $\rho \neq \varepsilon$ and (ii) $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \varphi$ (respectively, $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \psi'$), for all $\pi \in (\text{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \text{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. By Item (iv) of Proposition 3.1, we have that $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \psi$, for all $\pi \in (\text{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \text{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Hence, $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$.

(viii) (satisfiability constraint). If $\pi_1 \equiv_{\mathcal{K}}^{\psi} \pi_2$, then $\mathcal{K}, \rho \cdot \pi_{\geq 1} \models \psi$, for all $\pi \in (\text{Pth}(\mathcal{K}, \mathsf{lst}(\rho)) \cup \text{Trk}(\mathcal{K}, \mathsf{lst}(\rho)))$. Now, since there are two paths $\pi_1', \pi_2' \in \text{Pth}(\mathcal{K}, \mathsf{lst}(\rho))$ such that $\pi_1 = \rho \cdot (\pi_1')_{\geq 1}$ and $\pi_2 = \rho \cdot (\pi_2')_{\geq 1}$, we obtain that $\mathcal{K}, \pi_1 \models \psi$ and $\mathcal{K}, \pi_2 \models \psi$. □

At this point, we are able to prove that we can express the concept of tautology in GCTL itself, due to the particular structure formula equivalence chosen for the logic.

THEOREM 5.2 (STRUCTURE FORMULA TAUTOLOGY). *Let $\mathcal{K} = \langle \text{AP}, \text{W}, R, \mathsf{L}, w_0 \rangle$ be a Ks and $w \in \text{W}$ be one of its worlds. Moreover, let φ, φ_1, and φ_2 be state formulas and ψ be a path formula. Then, the following holds:*

(i) *φ is an $\cong_{\mathcal{K}}^{w}$-tautology if and only if $\mathcal{K}, w \models \varphi$;*
(ii) *$\mathsf{X}\, \psi$ cannot be an $\cong_{\mathcal{K}}^{w}$-tautology;*
(iii) *$\tilde{\mathsf{X}}\, \psi$ is an $\cong_{\mathcal{K}}^{w}$-tautology if and only if ψ is an $\cong_{\mathcal{K}}^{w'}$-tautology, for all $w' \in \text{W}$ such that $(w, w') \in R$;*
(iv) *$\varphi_1 \mathsf{U}\, \varphi_2$ is an $\cong_{\mathcal{K}}^{w}$-tautology if and only if $\mathcal{K}, w \models \varphi_2$;*
(v) *$\varphi_1 \mathsf{R}\, \varphi_2$ is an $\cong_{\mathcal{K}}^{w}$-tautology if and only if $\mathcal{K}, w \models \varphi_1 \wedge \varphi_2$;*
(vi) *$\varphi_1 \tilde{\mathsf{U}}\, \varphi_2$ is an $\cong_{\mathcal{K}}^{w}$-tautology if and only if $\mathcal{K}, w \models \mathsf{A}^{<1} \varphi_1 \tilde{\mathsf{U}}\, \varphi_2$;*
(vii) *$\varphi_1 \tilde{\mathsf{R}}\, \varphi_2$ is an $\cong_{\mathcal{K}}^{w}$-tautology if and only if $\mathcal{K}, w \models \mathsf{A}^{<1} \varphi_1 \tilde{\mathsf{R}}\, \varphi_2$.*

PROOF. We prove the statements case by case. In particular, note that we implicitly make use of properties of Proposition 3.1. Moreover, for Items (vi) and (vi), we only prove the (if) direction, since the converse is immediate by the definition of $\cong_{\mathcal{K}}^{w}$-equivalence.

(i) The thesis directly derives from the definition of $\cong_{\mathcal{K}}^{w}$-tautology.
(ii) The formula $\mathsf{X}\, \psi$ cannot be an $\cong_{\mathcal{K}}^{w}$-tautology, since $w \in \text{Trk}(\mathcal{K}, w)$ and $\mathcal{K}, w \not\models \mathsf{X}\, \psi$, where we remind that w for the path formula satisfiability relation $\models$ is considered as the track built only by the world w itself.
(iii) [Only if]. If $\tilde{\mathsf{X}}\, \psi$ is an $\cong_{\mathcal{K}}^{w}$-tautology, then $\mathcal{K}, \pi \models \tilde{\mathsf{X}}\, \psi$, for all $\pi \in (\text{Pth}(\mathcal{K}, w) \cup \text{Trk}(\mathcal{K}, w))$. Hence, we have that $\mathcal{K}, \pi_{\geq 1} \models \psi$, for all $\pi \in (\text{Pth}(\mathcal{K}, w) \cup \text{Trk}(\mathcal{K}, w))$ with $\pi_{\geq 1} \neq \varepsilon$, that is, $\pi \neq w$, which implies that $\mathcal{K}, \pi \models \psi$, for all $\pi \in (\text{Pth}(\mathcal{K}, w') \cup \text{Trk}(\mathcal{K}, w'))$ with $(w, w') \in R$. Hence, the thesis follows. *[If]*. The converse direction is perfectly specular to the previous one.
(iv) [Only if]. If $\varphi_1 \mathsf{U}\, \varphi_2$ is an $\cong_{\mathcal{K}}^{w}$-tautology, so is $\varphi_2 \vee \varphi_1 \wedge \mathsf{X}\, \varphi_1 \mathsf{U}\, \varphi_2$. Now, since $w \in \text{Trk}(\mathcal{K}, w)$, we have that $\mathcal{K}, w \models \varphi_2 \vee \varphi_1 \wedge \mathsf{X}\, \varphi_1 \mathsf{U}\, \varphi_2$ and so, $\mathcal{K}, w \models \varphi_2$, since $\mathcal{K}, w \not\models \mathsf{X}\, \varphi_1 \mathsf{U}\, \varphi_2$. *[If]*. If $\mathcal{K}, w \models \varphi_2$, then $\mathcal{K}, \pi \models \varphi_2 \vee \varphi_1 \wedge \mathsf{X}\, \varphi_1 \mathsf{U}\, \varphi_2$ and so $\mathcal{K}, \pi \models \varphi_1 \mathsf{U}\, \varphi_2$, for all $\pi \in (\text{Pth}(\mathcal{K}, w) \cup \text{Trk}(\mathcal{K}, w))$. Hence, $\varphi_1 \mathsf{U}\, \varphi_2$ is an $\cong_{\mathcal{K}}^{w}$-tautology.
(v) [Only if]. If $\varphi_1 \mathsf{R}\, \varphi_2$ is an $\cong_{\mathcal{K}}^{w}$-tautology, so is $\varphi_2 \wedge (\varphi_1 \vee \mathsf{X}\, \varphi_1 \mathsf{R}\, \varphi_2)$. Now, since $w \in \text{Trk}(\mathcal{K}, w)$, we have that $\mathcal{K}, w \models \varphi_2 \wedge (\varphi_1 \vee \mathsf{X}\, \varphi_1 \mathsf{R}\, \varphi_2)$ and so, $\mathcal{K}, w \models \varphi_1 \wedge \varphi_2$, since $\mathcal{K}, w \not\models \mathsf{X}\, \varphi_1 \mathsf{R}\, \varphi_2$. *[If]*. If $\mathcal{K}, w \models \varphi_1 \wedge \varphi_2$, then $\mathcal{K}, \pi \models \varphi_2 \wedge (\varphi_1 \vee \mathsf{X}\, \varphi_1 \mathsf{R}\, \varphi_2)$ and so $\mathcal{K}, \pi \models \varphi_1 \mathsf{R}\, \varphi_2$, for all $\pi \in (\text{Pth}(\mathcal{K}, w) \cup \text{Trk}(\mathcal{K}, w))$. Hence, $\varphi_1 \mathsf{R}\, \varphi_2$ is an $\cong_{\mathcal{K}}^{w}$-tautology.
(vi) By the hypothesis, we have that $\mathcal{K}, \pi \models \varphi_1 \tilde{\mathsf{U}}\, \varphi_2$, for all $\pi \in \text{Pth}(\mathcal{K}, w)$. Now, suppose by contradiction that $\varphi_1 \tilde{\mathsf{U}}\, \varphi_2$ is not an $\cong_{\mathcal{K}}^{w}$-tautology, that is, that there is a track $\rho \in \text{Trk}(\mathcal{K}, w)$ such that $\mathcal{K}, \rho \not\models \varphi_1 \tilde{\mathsf{U}}\, \varphi_2$. Then, we have that $\mathcal{K}, \rho \models (\neg\varphi_1) \mathsf{R}\, (\neg\varphi_2)$

and so $\mathcal{K}, \rho \models (\neg\varphi_2)\mathsf{U}(\neg\varphi_1 \wedge \neg\varphi_2)$, since ρ is necessarily finite. Now, consider a path $\pi \in \mathrm{Pth}(\mathcal{K}, w)$ having ρ as prefix, that is, such that $\pi_{\leq(|\rho|-1)} = \rho$. Then, it is evident that $\mathcal{K}, \pi \models (\neg\varphi_2)\mathsf{U}(\neg\varphi_1 \wedge \neg\varphi_2)$ and this implies that $\mathcal{K}, \pi \not\models \varphi_1 \tilde{\mathsf{U}} \varphi_2$, since there is no prefix in π satisfying φ_1 in all its positions before to reach a point in which φ_2 holds. Hence, we reached the contradiction.

(vii) By the hypothesis, we have that $\mathcal{K}, \pi \models \varphi_1 \tilde{\mathsf{R}} \varphi_2$, for all $\pi \in \mathrm{Pth}(\mathcal{K}, w)$. Now, suppose by contradiction that $\varphi_1 \tilde{\mathsf{R}} \varphi_2$ is not an $\cong^{w}_{\mathcal{K}}$-tautology, that is, that there exists a track $\rho \in \mathrm{Trk}(\mathcal{K}, w)$ such that $\mathcal{K}, \rho \not\models \varphi_1 \tilde{\mathsf{R}} \varphi_2$. Then, we have that $\mathcal{K}, \rho \models (\neg\varphi_1)\mathsf{U}(\neg\varphi_2)$. Now, consider a path $\pi \in \mathrm{Pth}(\mathcal{K}, w)$ having ρ as prefix, i.e., such that $\pi_{\leq(|\rho|-1)} = \rho$. Then, it is evident that $\mathcal{K}, \pi \models (\neg\varphi_1)\mathsf{U}(\neg\varphi_2)$ and this implies that $\mathcal{K}, \pi \not\models \varphi_1 \tilde{\mathsf{R}} \varphi_2$. Hence, we reached the contradiction. □

We now deduce two simple corollaries.

COROLLARY 5.1 (GCTL NEXT EQUIVALENCES). *Let* $\equiv_{\cdot}$ *be the prefix path equivalence. Moreover, let* φ *be a state formula and* $g \in [1, \omega]$. *Then, it holds that* $\mathsf{E}^{\geq g}\tilde{\mathsf{X}}\,\varphi \equiv \mathsf{E}\tilde{\mathsf{X}}(g, \varphi, \mathsf{E}^{\geq 1}\mathsf{X}\,\neg\varphi)$ *and* $\mathsf{A}^{<g}\mathsf{X}\,\varphi \equiv \mathsf{AX}(g, \varphi, \mathsf{A}^{<1}\tilde{\mathsf{X}}\,\neg\varphi)$.

PROOF. By Theorem 5.1, $\equiv_{\cdot}$ is adequate. Now, the thesis can be derived by Theorem 4.3 and Items i and iii of Theorem 5.2. □

In the rest of the article, we only consider formulas not containing any sub formula of the form $\mathsf{E}^{\geq g}\tilde{\mathsf{X}}\,\varphi$ with $\varphi \neq \mathfrak{f}$ and $\mathsf{A}^{<g}\mathsf{X}\,\varphi$ with $\varphi \neq \mathfrak{t}$. This can be done without loss of generality since each formula can be converted, with a linear blow-up only, into another one without these quantifications, by using the equivalence of the previous corollary.

COROLLARY 5.2 (GCTL FIXPOINT EQUIVALENCES). *Let* $\equiv_{\cdot}$ *be the prefix path equivalence. Moreover, let* φ_1 *and* φ_2 *be two state formulas and* $g \in [1, \omega]$. *Then, the following holds:*

(i) $\mathsf{E}^{\geq g}\varphi_1\mathsf{U}\,\varphi_2 \equiv \mathsf{EU}(g, \varphi_1, \varphi_2, \mathsf{E}^{\geq g}\varphi_1\mathsf{U}\,\varphi_2)$;
(ii) $\mathsf{E}^{\geq g}\varphi_1\mathsf{R}\,\varphi_2 \equiv \mathsf{ER}(g, \varphi_1, \varphi_2, \mathsf{E}^{\geq g}\varphi_1\mathsf{R}\,\varphi_2)$;
(iii) $\mathsf{E}^{\geq g}\varphi_1\tilde{\mathsf{U}}\,\varphi_2 \equiv \mathsf{E}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, \mathsf{E}^{\geq g}\varphi_1\tilde{\mathsf{U}}\,\varphi_2, \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq 1}\neg(\varphi_1\tilde{\mathsf{U}}\,\varphi_2))$;
(vi) $\mathsf{E}^{\geq g}\varphi_1\tilde{\mathsf{R}}\,\varphi_2 \equiv \mathsf{E}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, \mathsf{E}^{\geq g}\varphi_1\tilde{\mathsf{R}}\,\varphi_2, \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq 1}\neg(\varphi_1\tilde{\mathsf{R}}\,\varphi_2))$;
(v) $\mathsf{A}^{<g}\varphi_1\mathsf{U}\,\varphi_2 \equiv \mathsf{AU}(g, \varphi_1, \varphi_2, \mathsf{A}^{<g}\varphi_1\mathsf{U}\,\varphi_2, \mathsf{A}^{<1}\tilde{\mathsf{X}}\,\mathsf{A}^{<1}\neg(\varphi_1\mathsf{U}\,\varphi_2))$;
(vi) $\mathsf{A}^{<g}\varphi_1\mathsf{R}\,\varphi_2 \equiv \mathsf{AR}(g, \varphi_1, \varphi_2, \mathsf{A}^{<g}\varphi_1\mathsf{R}\,\varphi_2, \mathsf{A}^{<1}\tilde{\mathsf{X}}\,\mathsf{A}^{<1}\neg(\varphi_1\mathsf{R}\,\varphi_2))$;
(vii) $\mathsf{A}^{<g}\varphi_1\tilde{\mathsf{U}}\,\varphi_2 \equiv \mathsf{A}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, \mathsf{A}^{<g}\varphi_1\tilde{\mathsf{U}}\,\varphi_2)$;
(viii) $\mathsf{A}^{<g}\varphi_1\tilde{\mathsf{R}}\,\varphi_2 \equiv \mathsf{A}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, \mathsf{A}^{<g}\varphi_1\tilde{\mathsf{R}}\,\varphi_2)$.

PROOF. By Theorem 5.1, $\equiv_{\cdot}$ is adequate. Now, Items (i), (ii), (vii), and (viii) follow directly by Theorem 4.9, while Items (iii), (iv), (v), and (vi) can be derived by Theorem 4.10 and Items (iii), (vi), and (vii) of Theorem 5.2. □

We now conclude this part of the section by showing two simple but fundamental properties of GCTL* that allow the application of the automata-theoretic approach to the solution of the satisfiability problem for GCTL.

By using a proof by induction, we prove that GCTL* is invariant under the unwinding of a model.

THEOREM 5.3 (GCTL* UNWINDING INVARIANCE). *Let* $\equiv_{\cdot}$ *be the prefix path equivalence. Then,* GCTL* *is invariant w.r.t. unwinding, that is,* $\mathcal{K} \models \varphi$ *if and only if* $\mathcal{K}_U \models \varphi$, *for all state formulas* φ.

PROOF. Let $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, w_0\rangle$ be a Ks and $\mathcal{K}_U = \langle \mathrm{AP}, \mathrm{W}', R', \mathsf{L}', \varepsilon\rangle$ be its unwinding. Then, we show that for each GCTL* state formula φ and world $w \in \mathrm{W}'$, it holds that $\mathcal{K}, \mathsf{unw}(w) \models \varphi$ if and only if $\mathcal{K}_U, w \models \varphi$, where $\mathsf{unw} : \mathrm{W}' \to \mathrm{W}$ is the unwinding function. As a side result, we also prove that $\mathcal{K}, \mathsf{unw}(\pi) \models \psi$ if and only if $\mathcal{K}_U, \pi \models \psi$, for all

GCTL* path formulas ψ and paths/tracks $\pi \in (\mathrm{Pth}(\mathcal{K}_U, w) \cup \mathrm{Trk}(\mathcal{K}_U, w))$, where, in this case, $\mathsf{unw} : (\mathrm{Pth}(\mathcal{K}_U) \cup \mathrm{Trk}(\mathcal{K}_U)) \to (\mathrm{Pth}(\mathcal{K}) \cup \mathrm{Trk}(\mathcal{K}))$ is bijective function that extends the unwinding function on worlds to paths and tracks, that is, $(\mathsf{unw}(\pi))_i = \mathsf{unw}((\pi)_i)$, for all $i \in [0, |\pi|[$.

The proof proceeds by induction on the structure of the formula φ. The basic case of atomic propositions and the inductive cases of Boolean combinations are immediate and left to the reader. Therefore, let us consider the inductive case where φ is an existential quantification of the form $\mathsf{E}^{\geq g}\psi$, with $g \in [1, \omega]$. The case of universal quantifications $\mathsf{A}^{<g}\psi$ can be treated similarly.

First observe that, by the inductive hypothesis, it holds that $\mathcal{K}, \mathsf{unw}(w) \models \varphi$ if and only if $\mathcal{K}_U, w \models \varphi$, for all $\varphi \in \mathsf{rcl}(\psi)$ and $w \in \mathrm{W}'$. Now, it is immediate to see that $\mathcal{K}, \mathsf{unw}(\pi) \models \psi$ if and only if $\mathcal{K}_U, \pi \models \psi$, for all paths $\pi \in (\mathrm{Pth}(\mathcal{K}_U, w) \cup \mathrm{Trk}(\mathcal{K}_U, w))$. Indeed, by the definition of semantics on paths, we have that $\mathcal{K}, \mathsf{unw}(\pi) \models \psi$ if and only if $\varpi_{\mathcal{K},\psi}(\mathsf{unw}(\pi)) \models \psi$ and $\mathcal{K}_U, \pi \models \psi$ if and only if $\varpi_{\mathcal{K}_U,\psi}(\pi) \models \psi$. Now, by the previous observation and the definition of the path transformation, we have that $\varpi_{\mathcal{K},\psi}(\mathsf{unw}(\pi)) = \varpi_{\mathcal{K}_U,\psi}(\pi)$. Consequently, it holds that $\mathsf{unw}(\pi) \in \mathrm{Pth}(\mathcal{K}, \mathsf{unw}(w), \psi)$ if and only if $\pi \in \mathrm{Pth}(\mathcal{K}_U, w, \psi)$, for all $\pi \in \mathrm{Pth}(\mathcal{K}_U, w)$.

At this point, in order to prove that $|(\mathrm{Pth}(\mathcal{K}, \mathsf{unw}(w), \psi)/\equiv^{\psi}_{\mathcal{K}})| \geq g$ if and only if $|(\mathrm{Pth}(\mathcal{K}_U, w, \psi)/\equiv^{\psi}_{\mathcal{K}_U})| \geq g$, it remains to shows that $\pi_1 \equiv^{\psi}_{\mathcal{K}_U} \pi_2$ if and only if $\mathsf{unw}(\pi_1) \equiv^{\psi}_{\mathcal{K}} \mathsf{unw}(\pi_2)$. The case $\pi_1 = \pi_2$ is trivial. Thus, consider the case $\pi_1 \neq \pi_2$, let $\rho = \mathsf{pfx}(\pi_1, \pi_2)$ be their common prefix, and observe that $\mathsf{unw}(\rho) = \mathsf{pfx}(\mathsf{unw}(\pi_1), \mathsf{unw}(\pi_2))$. Now, by definition of prefix path equivalence, we have that $\pi_1 \equiv^{\psi}_{\mathcal{K}_U} \pi_2$ if and only if $\rho \neq \varepsilon$ and $\mathcal{K}_U, \rho \cdot \pi_{\geq 1} \models \psi$, for all $\pi \in (\mathrm{Pth}(\mathcal{K}_U, \mathsf{lst}(\rho)) \cup \mathrm{Trk}(\mathcal{K}_U, \mathsf{lst}(\rho)))$, and $\mathsf{unw}(\pi_1) \equiv^{\psi}_{\mathcal{K}} \mathsf{unw}(\pi_2)$ if and only if $\mathsf{unw}(\rho) \neq \varepsilon$ and $\mathcal{K}, \mathsf{unw}(\rho) \cdot \pi'_{\geq 1} \models \psi$, for all $\pi' \in (\mathrm{Pth}(\mathcal{K}, \mathsf{lst}(\mathsf{unw}(\rho))) \cup \mathrm{Trk}(\mathcal{K}, \mathsf{lst}(\mathsf{unw}(\rho))))$. Now, using again the fact that $\mathcal{K}, \mathsf{unw}(\pi) \models \psi$ if and only if $\mathcal{K}_U, \pi \models \psi$, for all paths/tracks $\pi \in (\mathrm{Pth}(\mathcal{K}_U, w) \cup \mathrm{Trk}(\mathcal{K}_U, w))$, the thesis follows. □

Directly from the previous result, we obtain that GCTL* also enjoys the tree model property.

COROLLARY 5.3 (GCTL* TREE MODEL PROPERTY). *Let $\equiv^{\cdot}_{\cdot}$ be the prefix path equivalence. Then,* GCTL* *has the* tree model property.

PROOF. Consider a formula φ and suppose that it is satisfiable. Then, there is a KS $\mathcal{K}$ such that $\mathcal{K} \models \varphi$. By Theorem 5.3, φ is satisfied at the root of the unwinding $\mathcal{K}_U$ of $\mathcal{K}$. Thus, since $\mathcal{K}_U$ is a KT, we immediately have that φ is satisfied on a tree model. □

5.2. GCTL vs GμCalculus Relationships

The μCALCULUS [Kozen 1983] is a well-known modal logic augmented with fixed point operators, which subsumes the classical temporal logics such as LTL, CTL, and CTL*. The GμCALCULUS simply extends the μCALCULUS with graded state quantifiers [Kupferman et al. 2002; Bonatti et al. 2008].

In the next theorem, we show a double-exponential reduction from the significant fragment of GCTL without infinite-degree quantifications to GμCALCULUS.

THEOREM 5.4 (GCTL-GμCALCULUS REDUCTION). *For each* GCTL *formula φ free of the $\mathsf{E}^{\geq\omega}$ and $\mathsf{A}^{<\omega}$ quantifications, it is possible to construct an equisatisfiable formula χ of* GμCALCULUS *with $\|\chi\| = \mathrm{O}(2^{k \cdot \sqrt{\hat{\varphi}} \cdot |\varphi|})$, for a constant k, that is, φ is satisfiable if and only if χ is satisfiable.*

PROOF. The reduction we now propose is almost a translation by equivalence. The only basic formulas that cannot be directly translated are the quantifications $\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f}$

and $\mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t}$ that are satisfied, respectively, only on worlds without and with successors. This is because the μCALCULUS, and so the GμCALCULUS, is usually defined only on total Ks, and $\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f}$ and $\mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t}$ are equivalent to $\mathfrak{f}$ and $\mathfrak{t}$, respectively, on such a kind of structures. To overcome this gap, we enrich each Ks with a fresh atomic proposition *end*, representing the fact that a world has no successors, and translate $\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f}$ in *end* and $\mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t}$ in $\neg$*end*. Moreover, we force the translation of (i) $\mathsf{E}^{\geq g}\mathsf{X}\,\varphi$ to ensure that it is satisfied only on worlds not labeled with *end* and (ii) $\mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi$ to allow that it is satisfied also on worlds labeled with *end*, where $g \in [1, \omega[$. Apart from the cases of the atomic propositions, the Boolean connectives, and the quantifiers $\mathsf{E}^{\geq 0}\psi$ and $\mathsf{A}^{<0}\psi$ that are equivalent to $\mathfrak{t}$ and $\mathfrak{f}$, respectively, the remaining case are solved using the equivalence showed in Corollary 5.2. Formally, the translation $\chi = \overline{\varphi}$ of φ is inductively defined as follows, where $g \in [1, \omega[$:

(1) $\overline{p} \triangleq p$, for $p \in \mathrm{AP}$;
(2) $\overline{\neg\varphi} \triangleq \neg\,\overline{\varphi}$; $\quad \overline{\varphi_1 \wedge \varphi_2} \triangleq \overline{\varphi_1} \wedge \overline{\varphi_2}$; $\quad \overline{\varphi_1 \vee \varphi_2} \triangleq \overline{\varphi_1} \vee \overline{\varphi_2}$;
(3) $\overline{\mathsf{E}^{\geq 0}\psi} \triangleq \mathfrak{t}$; $\quad \overline{\mathsf{A}^{<0}\psi} \triangleq \mathfrak{f}$;
(4) $\overline{\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f}} \triangleq end$; $\quad \overline{\mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t}} \triangleq \neg end$;
(5) $\overline{\mathsf{E}^{\geq g}\mathsf{X}\,\varphi} \triangleq \neg\, end \wedge \langle g-1\rangle\,\overline{\varphi}$; $\quad \overline{\mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi} \triangleq end \vee [g-1]\,\overline{\varphi}$;
(6) $\overline{\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2)} \triangleq \mu Y.\overline{\mathsf{EU}(g, \varphi_1, \varphi_2, Y)}$;
(7) $\overline{\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2)} \triangleq \nu Y.\overline{\mathsf{ER}(g, \varphi_1, \varphi_2, Y)}$;
(8) $\overline{\mathsf{E}^{\geq g}(\varphi_1 \tilde{\mathsf{U}}\, \varphi_2)} \triangleq \mu Y.\overline{\mathsf{E}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, Y, \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq 1}\neg(\varphi_1 \tilde{\mathsf{U}}\, \varphi_2))}$;
(9) $\overline{\mathsf{E}^{\geq g}(\varphi_1 \tilde{\mathsf{R}}\, \varphi_2)} \triangleq \nu Y.\overline{\mathsf{E}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, Y, \mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq 1}\neg(\varphi_1 \tilde{\mathsf{R}}\, \varphi_2))}$;
(10) $\overline{\mathsf{A}^{<g}(\varphi_1 \mathsf{U}\, \varphi_2)} \triangleq \mu Y.\overline{\mathsf{AU}(g, \varphi_1, \varphi_2, Y, \mathsf{A}^{<1}\tilde{\mathsf{X}}\,\mathsf{A}^{<1}\neg(\varphi_1 \mathsf{U}\, \varphi_2))}$;
(11) $\overline{\mathsf{A}^{<g}(\varphi_1 \mathsf{R}\, \varphi_2)} \triangleq \nu Y.\overline{\mathsf{AR}(g, \varphi_1, \varphi_2, Y, \mathsf{A}^{<1}\tilde{\mathsf{X}}\,\mathsf{A}^{<1}\neg(\varphi_1 \mathsf{R}\, \varphi_2))}$;
(12) $\overline{\mathsf{A}^{<g}(\varphi_1 \tilde{\mathsf{U}}\, \varphi_2)} \triangleq \mu Y.\overline{\mathsf{A}\tilde{\mathsf{U}}(g, \varphi_1, \varphi_2, Y)}$;
(13) $\overline{\mathsf{A}^{<g}(\varphi_1 \tilde{\mathsf{R}}\, \varphi_2)} \triangleq \nu Y.\overline{\mathsf{A}\tilde{\mathsf{R}}(g, \varphi_1, \varphi_2, Y)}$.

By induction on the structure of the formula, it is not hard to see that, for each Ks $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, w_0\rangle$ model of φ, the Ks $\mathcal{K}' = \langle \mathrm{AP} \cup \{end\}, \mathrm{W}, R', \mathsf{L}', w_0\rangle$ is a model of $\overline{\varphi}$, where (i) $R' \cap (\mathrm{W} \setminus \mathrm{W}') \times \mathrm{W} = R$, (ii) $\mathsf{L}'(w) = \mathsf{L}(w)$, (iii) $\mathsf{L}'(w') = \mathsf{L}(w') \cup \{end\}$, and (iv) $(w', w') \in R'$, for all $w \in \mathrm{W} \setminus \mathrm{W}'$ and $w' \in \mathrm{W}'$, with $\mathrm{W}' = \{w \in \mathrm{W} : \nexists w' \in \mathrm{W}.(w, w') \in R\}$. Intuitively, we simply add to each world having no successors a self loop and the label *end*. Moreover, from a Ks $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, R, \mathsf{L}, w_0\rangle$ model of $\overline{\varphi}$, it is possible to extract a Ks $\mathcal{K}' = \langle \mathrm{AP}, \mathrm{W}, R', \mathsf{L}, w_0\rangle$ model of φ, by simply substituting the transition relation R with a new relation R' defined as follows: $(w, w') \in R'$ if and only if $(w, w') \in R$ and $end \notin \mathsf{L}(w)$, for all $w, w' \in \mathrm{W}$. Intuitively, we simply cut out each edge exiting from a world labeled with *end*.

Finally, we turn to the size of χ. First, note that all points 1–5 are linear. Instead, points 6–13 are exponential in the degree of the original formula, and so, double exponential in its size, since they are based on the expansion formulas of Theorems 4.9 and 4.10. With more details, each of these transformations give a blow-up that is an $\mathsf{O}((|\overline{\varphi_1}| + |\overline{\varphi_2}|) \cdot 2^{k\cdot\sqrt{g}})$. Now, by a simple calculation, since the nesting of such a kind of formulas is bounded by the length of φ, we obtain that $\|\chi\| = \mathsf{O}(2^{k\cdot\sqrt{\mathring{\varphi}}\cdot|\varphi|})$. It is important to remark that the number of disjunctions in χ can be exponential in the degree $\mathring{\varphi}$. Therefore, even using a DAG to represent χ, it would only reduce the overall size to $\mathsf{O}(|\varphi| \cdot 2^{k\cdot\sqrt{\mathring{\varphi}}})$. Hence, it remains double exponential in $\|\varphi\|$. □

By the previous theorem and the fact that for GμCALCULUS the satisfiability problem is solvable in EXPTIME [Kupferman et al. 2002], we immediately get that the problem for the given fragment of GCTL is decidable and solvable in 3ExpTime. However, in the next chapters we improve this result by showing that the problem for the whole GCTL is solvable in ExpTime, by exploiting an automata-theoretic approach.

Finally, we show that GCTL is at least exponentially more succinct than GμCalculus, both with the binary coding of the degree. We prove the statement by showing a class of GCTL formulas φ_g, with $g \in [1, \omega[$, whose minimal equivalent GμCalculus formulas χ_g needs to be, in size, exponentially bigger than (the size of) φ_g. Classical techniques [Lange 2008; Lutz 2006; Wilke 1999] rely on the fact that in the more succinct logic there exists a formula having a *least finite model* whose size is double exponential in the size of the formula, while in the less succinct logic every satisfiable formula has finite models of size at most exponential in its size. Unfortunately, in our case we cannot apply this idea, since, as far as we know, both GCTL and the GμCalculus satisfy the small model property, that is, all their satisfiable formulas have always a model at most exponential in their size. Hence, to prove the succinctness of GCTL, we explore a technique based on a characteristic property of our logic. Specifically, it is based on the fact that, using GCTL, we can write a set of formulas φ_g each one having a number of "characterizing models" that is exponential in the degree g of φ_g, while every GμCalculus formula has at most a polynomial number of those models in its degree.

Consider the property "in a tree, there are exactly g grandchildren of the root having only one path leading from them, and these grandchildren are all and only the nodes labeled with p". Such a property can be easily described by the GCTL formula $\varphi_g = \varphi' \wedge \varphi''_g$, where $\varphi' = \neg p \wedge \mathsf{A}^{<1}\mathsf{X}\,(\neg p \wedge \mathsf{A}^{<1}\mathsf{X}\,(p \wedge \mathsf{A}^{<1}\mathsf{X}\,\mathsf{A}^{<1}\mathsf{G}\,(\neg p \wedge \mathsf{A}^{<2}\tilde{\mathsf{X}}\,\mathfrak{f})))$ and $\varphi''_g = \mathsf{E}^{=g}\mathsf{F}\,p$. By simple a calculation, we can see that $|\varphi_g| = 31$, $\mathring{\varphi}_g = g$, and $\|\varphi_g\| = 32 + \lceil\log(g)\rceil + \lceil\log(g+1)\rceil$. So, its size is $\Theta(\lceil\log(g)\rceil)$. We claim that a G$\mu$Calculus formula χ_g requires exponential size to express the same property. More formally, our aim is to prove the following theorem.

Theorem 5.5 (GCTL Exponential Succinctness). *Let $\varphi_g = \varphi' \wedge \varphi''_g$, with $\varphi' = \neg p \wedge \mathsf{A}^{<1}\mathsf{X}\,(\neg p \wedge \mathsf{A}^{<1}\mathsf{X}\,(p \wedge \mathsf{A}^{<1}\mathsf{X}\,\mathsf{A}^{<1}\mathsf{G}\,(\neg p \wedge \mathsf{A}^{<2}\,\tilde{\mathsf{X}}\,\mathfrak{f})))$, $\varphi''_g = \mathsf{E}^{=g}\mathsf{F}\,p$, and $g \in [1, \omega[$. Then, each* GμCalculus *formula χ_g equivalent to φ_g has size $\Omega(2^{\|\varphi_g\|})$.*

The proof of this theorem proceeds directly by proving the following lemma and observing that, since $\|\varphi_g\| = \Theta(\lceil\log(g)\rceil)$, we can easily derive that $\|\chi_g\| = \Omega(2^{\|\varphi_g\|})$.

Lemma 5.1 (GμCalculus Polynomial Degree Lower Bound). *For all the* GμCalculus *formulas χ_g equivalent to φ_g, it holds that they have size $\Omega(g)$.*

Proof. To prove this, we use an automata-theoretic approach. We first recall that the automata model developed in Kupferman et al. [2002], used to accept all and only the tree models of a GμCalculus formula χ, has as set of states the closure set of χ. On every accepting run, when the automaton is in a state q on a node x of the input tree, the subtree rooted at that node is a model of q. Our aim now is to prove that the automaton $\mathcal{A}_{\chi_g}$ for χ_g can accept all and only the models of χ_g, and so of φ_g, only if its state space contains either a formula $\langle i\rangle\phi$ or a formula $[i]\phi$, for all $i \in [0, g[$. Recall that the GμCalculus formulas $\langle i\rangle\phi$ and $[i]\phi$ mean that there are at least $i+1$ successor satisfying ϕ and all but at most i successors satisfy ϕ, respectively. Suppose by contradiction that there is no formula ϕ such that $\langle i\rangle\phi$ or $[i]\phi$ are in the state space of $\mathcal{A}_{\chi_g}$, for a given index i. Since $\mathcal{A}_{\chi_g}$ accepts all the models of φ_g, it accepts the input tree $\mathcal{T} = \langle \mathsf{T}, \mathsf{v}\rangle$, where $\mathsf{T} = \{\varepsilon\} \cup \{0, 1\} \cup \{0 \cdot 0 \cdot 0^*, \ldots, 0 \cdot (i-1) \cdot 0^*, 1 \cdot 0 \cdot 0^*, \ldots, 1 \cdot (g-i-1) \cdot 0^*\}$, every node x, with $|x| = 2$, that is, of level equal to 2, is labeled with $\mathsf{v}(x) = \{p\}$, and every other node y is labeled with $\mathsf{v}(y) = \emptyset$. Informally, node 0 has i successors labeled with p, while node 1 has $g - i$ successors labeled in the same way. Now, on the accepting run $\mathcal{R}$ of $\mathcal{A}_{\chi_g}$ on $\mathcal{T}$ in the node 0, the active states represent what are needed to be satisfied in the current node and such requirements do no contain any existential $\langle i \rangle \phi$ or universal $[i]\phi$. Hence, if we substitute $\mathcal{T}$ with a new tree $\mathcal{T}'$ having only $i - 1$ successor of 0 (labeled with p), then we obtain that also $\mathcal{T}'$ is accepted, reaching in this way the contradiction. This is due to the fact that, we can easily modify the run $\mathcal{R}$ to construct an accepting run $\mathcal{R}'$ for $\mathcal{T}'$, by removing all its subtrees rooted at a node whose label contains the node $0 \cdot l$, with $l \in [0, g[$, not in $\mathcal{T}'$. Indeed, when $\mathcal{A}_{\chi_g}$ is on the node 0, every nonquantified formula is already satisfied. A formula $\langle j \rangle \phi$ with $j > i$ could not be required on $\mathcal{T}$, and so on $\mathcal{T}'$, since it would be trivially false anyway. A formula $[j]\phi$ with $j > i$ is trivially true on both the trees. Finally, formulas $\langle j \rangle \phi$ or $[j]\phi$, with $j < i$, are satisfied on $\mathcal{T}$ by hypothesis. Now, since the subtrees rooted at the successor nodes of 0 are all equal, they all satisfy ϕ. Thus, by removing one of them, the quantifier formula is still satisfied. This reasoning shows that the closure of χ_g contains at least an existential or universal formula for each degree $i \in [0, g[$. Hence, the formula χ_g must have at least size $\Omega(g)$. □

Note that, as far as we know, the size of the smallest GμCALCULUS formula χ equivalent to φ has size double exponential in the binary coding of the degree g. In particular, χ can be obtained by using the translation $\overline{\varphi}$ described in Theorem 5.4. So, there is an exponential gap between upper and lower bound for the translation from GCTL to GμCALCULUS. Actually, we conjecture that the succinctness is tight for double exponential, but the technique used in the previous lemma does not seem to be adaptable for a double exponential lower bound.

6. ALTERNATING TREE AUTOMATA

In this section, we briefly introduce an automaton model used to solve efficiently the satisfiability problems for GCTL in EXPTIME w.r.t. the size of the formula, by reducing this problem to the emptiness of the automaton. We recall that, in general, an approach with tree automata to the solution of the satisfiability problem is only possible once the logic satisfies the tree model property. In fact, this property holds for GCTL*, and consequently for GCTL, as we have proved in Corollary 5.3.

6.1. Classic Automata

Nondeterministic tree automata are a generalization to infinite trees of the classical *nondeterministic word automata* (see Thomas [1990], for an introduction). *Alternating tree automata* are a further generalization of nondeterministic tree automata [Muller and Schupp 1987]. Intuitively, on visiting a node of the input tree, while the latter sends exactly one copy of itself to each of the successors of the node, the first can send several copies of itself to the same successor.

We now give the formal definition of alternating tree automata.

Definition 6.1 (*Alternating Tree Automata*). An *alternating tree automaton* (ATA, for short) is a tuple $\mathcal{A} \triangleq \langle \Sigma, \Delta, \mathrm{Q}, \delta, q_0, \mathrm{F} \rangle$, where Σ, Δ, and Q are nonempty finite sets of *input symbols*, *directions*, and *states*, respectively, $q_0 \in \mathrm{Q}$ is an *initial state*, F is an *acceptance condition* to be defined later, and $\delta : \mathrm{Q} \times \Sigma \to \mathrm{B}^+(\Delta \times \mathrm{Q})$ is an *alternating transition function* that maps each pair of states and input symbols to a positive Boolean combination on the set of propositions of the form $(d, q) \in \Delta \times \mathrm{Q}$, a.k.a. *moves*.

A nondeterministic tree automaton (NTA) is a special ATA in which each conjunction in the transition function δ has exactly one move (d, q) associated with each direction d. In addition, a *universal tree automaton* (UTA) is a special ATA in which all the Boolean combinations that appear in δ are only conjunctions of moves.

The semantics of the ATAS is now given through the following concept of run.

Definition 6.2 (ATA *Run*). A *run* of an ATA $\mathcal{A} = \langle \Sigma, \Delta, Q, \delta, q_0, F \rangle$ on a Σ-labeled Δ-tree $\mathcal{T} = \langle T, v \rangle$ is a $(Q \times T)$-labeled $\mathbb{N}$-tree $\mathcal{R} \triangleq \langle R, r \rangle$ such that (i) $r(\varepsilon) = (q_0, \varepsilon)$ and (ii) for all nodes $y \in R$ with $r(y) = (q, x)$, there is a set of moves $S \subseteq \Delta \times Q$ with $S \models \delta(q, v(x))$ such that, for all $(d, q') \in S$, there is an index $j \in [0, |S|[$ for which it holds that $y \cdot j \in R$ and $r(y \cdot j) = (q', x \cdot d)$.

In the following, we only consider ATAs along with the *parity* acceptance condition (APT, for short) $F = (F_1, \ldots, F_k) \in (2^Q)^+$ with $F_1 \subseteq \ldots \subseteq F_k = Q$ (see Kupferman et al. [2000], for more). The number k of sets in F is called the *index* of the automaton.

Let $\mathcal{R} = \langle R, r \rangle$ be a run of an ATA $\mathcal{A}$ on a tree $\mathcal{T} = \langle T, v \rangle$ and $R' \subseteq R$ one of its branches. Then, by $\mathsf{inf}(R') \triangleq \{q \in Q : |\{y \in R' : \exists x \in T. r(y) = (q, x)\}| = \omega\}$ we denote the set of states that occur infinitely often as labeling of the nodes in the branch R'. We say that a branch R' of $\mathcal{T}$ satisfies the parity acceptance condition $F = (F_1, \ldots, F_k)$ if and only if the least index $i \in [1, k]$ for which $\mathsf{inf}(R') \cap F_i \neq \emptyset$ is even.

At this point, we can define the concept of language accepted by an ATA.

Definition 6.3 (ATA *Acceptance*). An ATA $\mathcal{A} = \langle \Sigma, \Delta, Q, \delta, q_0, F \rangle$ *accepts* a Σ-labeled Δ-tree $\mathcal{T}$ if and only if there exists a run $\mathcal{R}$ of $\mathcal{A}$ on $\mathcal{T}$ such that all its infinite branches satisfy the acceptance condition F, where the concept of satisfaction is dependent from the definition of F.

By $L(\mathcal{A})$ we denote the language accepted by the ATA $\mathcal{A}$, that is, the set of trees $\mathcal{T}$ accepted by $\mathcal{A}$. Moreover, $\mathcal{A}$ is said to be *empty* if $L(\mathcal{A}) = \emptyset$. The *emptiness problem* for $\mathcal{A}$ is to decide whether $L(\mathcal{A}) = \emptyset$ or not.

6.2. Automata with Satellite

As a generalization of ATA, here we consider *alternating tree automata with satellites* (ATAS), in a similar way it has been done in Kupferman and Vardi [2006], with the main difference that our satellites are nondeterministic and can work on trees and not only on words. The satellite is used to ensure that the input tree satisfies some structural properties and it is kept apart from the main automaton as it allows to show a tight complexity for the satisfiability problems.

We now formally define this new fundamental concept of automaton.

Definition 6.4 (*Alternating Tree Automata with Satellite*). An *alternating tree automaton with satellite* (ATAS, for short) is a tuple $\langle \mathcal{A}, \mathcal{S} \rangle$, where $\mathcal{A} \triangleq \langle \Sigma \times P_E, \Delta, Q, \delta, q_0, F \rangle$ is an ATA and $\mathcal{S} \triangleq \langle \Sigma, \Delta, P, \zeta, P_0 \rangle$ is a *nondeterministic safety automaton*, a.k.a. *satellite*, where $P = P_E \times P_I$ is a nonempty finite set of *states* split in two components, *external* P_E and *internal* P_I states, $P_0 \subseteq P$ is a set of *initial states*, and $\zeta : P \times \Sigma \to 2^{P^{\Delta}}$ is a *nondeterministic transition function* that maps a state and an input symbol to a set of functions from directions to states. The set Σ is the *alphabet* of the ATAS $\langle \mathcal{A}, \mathcal{S} \rangle$.

The semantics of satellites is given through the following concepts of run, acceptance, and building. It is possible to note a similarity with the concept of cascade product automata that can be found in literature.

Definition 6.5 (*Satellite Run*). A *run* of a satellite $\mathcal{S} = \langle \Sigma, \Delta, P, \zeta, P_0 \rangle$ on a Σ-labeled Δ-tree $\mathcal{T} = \langle T, v \rangle$ is a P-labeled Δ-tree $\mathcal{R} \triangleq \langle T, r \rangle$ such that (i) $r(\varepsilon) \in P_0$ and (ii) for all nodes $x \in T$ with $r(x) = p$, there is a function $g \in \zeta(p, v(x))$ such that, for all $d \in \Delta$ with $x \cdot d \in T$, it holds that $r(x \cdot d) = g(d)$.

Definition 6.6 (*Satellite Acceptance*). A satellite $\mathcal{S} = \langle \Sigma, \Delta, P, \zeta, P_0 \rangle$ *accepts* a Σ-labeled Δ-tree $\mathcal{T}$ if and only if there exists a run $\mathcal{R}$ of $\mathcal{S}$ on $\mathcal{T}$.

For the coming definition we have to introduce an extra notation. Given a $(\Sigma' \times \Sigma'')$-labeled Δ-tree $\mathcal{T} = \langle \mathrm{T}, \mathsf{v} \rangle$, we define the *projection* of $\mathcal{T}$ on Σ' as the Σ'-labeled Δ-tree $\mathcal{T}_{\downarrow \Sigma'} \triangleq \langle \mathrm{T}, \mathsf{v}' \rangle$ such that, for all nodes $x \in \mathrm{T}$, we have $\mathsf{v}(x) = (\mathsf{v}'(x), \sigma)$, for some $\sigma \in \Sigma''$. Moreover, given a Σ'-labeled Δ-tree $\mathcal{T}' = \langle \mathrm{T}, \mathsf{v}' \rangle$ and a Σ''-labeled Δ-tree $\mathcal{T}'' = \langle \mathrm{T}, \mathsf{v}'' \rangle$, we define the *combination* of $\mathcal{T}'$ with $\mathcal{T}''$ as the $(\Sigma' \times \Sigma'')$-labeled Δ-tree $\mathcal{T}' \otimes \mathcal{T}'' \triangleq \langle \mathrm{T}, \mathsf{v} \rangle$ such that, for all nodes $x \in \mathrm{T}$, we have $\mathsf{v}(x) = (\mathsf{v}'(x), \mathsf{v}''(x))$.

Definition 6.7 (*Satellite Building*). A satellite $\mathcal{S} = \langle \Sigma, \Delta, \mathrm{P}, \zeta, \mathrm{P}_0 \rangle$ with $\mathrm{P} = \mathrm{P}_E \times \mathrm{P}_I$ *builds* a $\Sigma \times \mathrm{P}_E$-labeled Δ-tree $\mathcal{T}_{\mathcal{S}}$ over a Σ-labeled Δ-tree $\mathcal{T}$ if and only if there exists a run $\mathcal{R}$ of $\mathcal{S}$ on $\mathcal{T}$ such that $\mathcal{T}_{\mathcal{S}}$ is the combination $\mathcal{T} \otimes \mathcal{R}_{\downarrow \mathrm{P}_E}$ of $\mathcal{T}$ with the projection of $\mathcal{R}$ on P_E.

At this point, we can define the language accepted by an AtaS.

Definition 6.8 (AtaS *Acceptance*). A Σ-labeled Δ-tree $\mathcal{T}$ is accepted by an AtaS $\langle \mathcal{A}, \mathcal{S} \rangle$, where $\mathcal{A} = \langle \Sigma \times \mathrm{P}_E, \Delta, \mathrm{Q}, \delta, q_0, \mathrm{F} \rangle$, $\mathcal{S} = \langle \Sigma, \Delta, \mathrm{P}, \zeta, \mathrm{P}_0 \rangle$, and $\mathrm{P} = \mathrm{P}_E \times \mathrm{P}_I$, if and only if $\mathcal{S}$ builds a tree $\mathcal{T}_{\mathcal{S}}$ over $\mathcal{T}$ such that $\mathcal{T}_{\mathcal{S}}$ is accepted by the Ata $\mathcal{A}$.

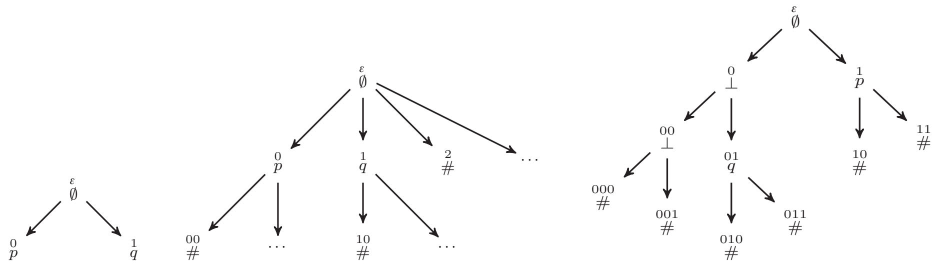
In words, first the satellite $\mathcal{S}$ guesses and adds to the input tree $\mathcal{T}$ an additional labeling over the set P_E, thus returning the built tree $\mathcal{T}_{\mathcal{S}}$. Then, the main automaton $\mathcal{A}$ computes a new run on $\mathcal{T}_{\mathcal{S}}$ taken as input. By $\mathrm{L}(\langle \mathcal{A}, \mathcal{S} \rangle)$ we denote the language accepted by the AtaS $\langle \mathcal{A}, \mathcal{S} \rangle$.

In the following, we consider, in particular, AtaS along with the parity acceptance condition (AptS, for short).

Note that satellites are just a convenient way to describe an Ata in which the state space can be partitioned into two components, one of which is nondeterministic, independent from the other, and that has no influence on the acceptance. Indeed, it is just a matter of technicality to see that automata with satellites inherit all the closure properties of alternating automata. In particular, the following theorem, directly derived by a proof idea of Kupferman and Vardi [2006], shows how the separation between $\mathcal{A}$ and $\mathcal{S}$ gives a tight analysis of the complexity of the relative emptiness problem.

Theorem 6.1 (AptS Emptiness). *The* emptiness problem *for an* AptS $\langle \mathcal{A}, \mathcal{S} \rangle$ *with alphabet size h, where the main automaton $\mathcal{A}$ has n states and index k and the satellite $\mathcal{S}$ has m states, can be decided in time* $2^{\mathrm{O}(\log(h)+(n \cdot k)\cdot((n \cdot k)\cdot\log(n \cdot k)+\log(m)))}$.

Proof. As first thing, we use the Muller-Schupp exponential-time nondeterminization procedure [Muller and Schupp 1995] that leads from the Apt $\mathcal{A}$ to an Npt $\mathcal{N}$, with $2^{\mathrm{O}((n \cdot k)\cdot\log(n \cdot k))}$ states and index $\mathrm{O}(n \cdot k)$, such that $\mathrm{L}(\mathcal{A}) = \mathrm{L}(\mathcal{N})$. Since an Npt is a particular Apt, we immediately have that $\mathrm{L}(\langle \mathcal{N}, \mathcal{S} \rangle) = \mathrm{L}(\langle \mathcal{A}, \mathcal{S} \rangle)$. At this point, by taking the product-automaton between $\mathcal{N}$ and the satellite $\mathcal{S}$, we obtain another Npt $\mathcal{N}^\star$, with $2^{\mathrm{O}((n \cdot k)\cdot\log(n \cdot k)+\log(m))}$ states and index $\mathrm{O}(n \cdot k)$, such that $\mathrm{L}(\mathcal{N}^\star) = \mathrm{L}(\langle \mathcal{N}, \mathcal{S} \rangle)$. With more details, if $\mathcal{N} = \langle \Sigma \times \mathrm{P}_E, \Delta, \mathrm{Q}, \delta, \mathrm{Q}_0, \mathrm{F} \rangle$ and $\mathcal{S} = \langle \Sigma, \Delta, \mathrm{P}, \zeta, \mathrm{P}_0 \rangle$ with $\mathrm{P} = \mathrm{P}_E \times \mathrm{P}_I$ and $\mathrm{F} = (\mathrm{F}_1, \ldots, \mathrm{F}_k)$, we have that $\mathcal{N}^\star \triangleq \langle \Sigma, \Delta, \mathrm{Q} \times \mathrm{P}, \delta^\star, \mathrm{Q}_0 \times \mathrm{P}_0, \mathrm{F}^\star \rangle$ with $\mathrm{F}^\star \triangleq (\mathrm{F}_1 \times \mathrm{P}, \ldots, \mathrm{F}_k \times \mathrm{P})$ and $\delta^\star((q, (p_E, p_I)), \sigma) \triangleq (\bigvee_{\mathsf{g} \in \zeta((p_E, p_I), \sigma)} \delta(q, (\sigma, p_E)))[(d, q') \in \Delta \times \mathsf{Q}/(d, (q', \mathsf{g}(d)))]$, where by $f[x/y]$ we denote the formula in which all occurrences of a proposition x in f are replaced by the proposition y. In words, $\delta^\star((q, (p_E, p_I)), \sigma)$ is obtained by guessing what is the choice g of the satellite in the state (p_E, p_I) when it reads σ and then by substituting in $\delta(q, (\sigma, p_E))$ each occurrence of a move (d, q') with a new move of the form $(d, (q', p'))$, where $p' = \mathsf{g}(d)$ represents the new state sent by the satellite in the direction d. Hence, it is evident that $\mathrm{L}(\mathcal{N}^\star) = \mathrm{L}(\langle \mathcal{A}, \mathcal{S} \rangle)$ by definition of AtaS. Now, the emptiness of $\mathcal{N}^\star$ can be checked in polynomial running-time in its number of states, exponential in its index, and linear in the alphabet size (see Theorem 5.1 of Kupferman



(a) $\mathcal{T}$. (b) $\mathcal{T}_W$. (c) $\mathcal{T}_D$.

Fig. 4. A tree $\mathcal{T}$, its widening $\mathcal{T}_W$, and the related delayed generation $\mathcal{T}_D$.

and Vardi [1998]). Overall, with this procedure, we obtain that the emptiness problem for an AptS is solvable in time $2^{O(\log(h)+(n\cdot k)\cdot((n\cdot k)\cdot\log(n\cdot k)+\log(m)))}$. □

7. GCTL MODEL TRANSFORMATIONS

At this point, we can start to describe the decision procedure for the satisfiability problem of GCTL. As we discussed in the introduction, we exploit an automata-theoretic approach by using satellites that are able to accept binary tree-encodings of tree models of a formula. So, we first introduce the binary tree encoding and then, in the next section, we show how to build the automaton accepting all tree-model encodings satisfying the formula of interest.

The tree encoding works as follows. Given a tree model $\mathcal{T}$ of φ, we first build its *widening* $\mathcal{T}_W$, obtaining in this way a full tree with infinite branching. Then, from $\mathcal{T}_W$, we derive a *delayed generation* tree $\mathcal{T}_D$ that embeds $\mathcal{T}_W$ in a binary tree. Finally, we enrich the labeling of $\mathcal{T}_W$ with degree functions that allow to propagate the information related to the degree g of the formula along the paths. This is done by using a set B of elements, called bases, that are used in the domain of the degree functions. The obtained tree $\mathcal{T}_{D_{B,g}}$ is named B-*based g-degree delayed generation*. Intuitively, a base is used to represent a subformula of φ to which we associate, by means of the degree functions, the related number of paths required to be satisfied. This turns to be a key step in the whole satisfiability procedure we show in the next section.

In the following, to simplify the technical reasoning, we use as unwinding of a Ks $\mathcal{K}$, not the Kt $\mathcal{K}_U$ itself, but one of the complete 2^{AP}-labeled $\mathbb{N}$-tree $\mathcal{T}$ isomorphic to $\mathcal{K}_U$.

7.1. Binary Tree Model Encoding

As first step in our binary encoding construction, we define the widening of a 2^{AP}-labeled $\mathbb{N}$-tree $\mathcal{T}$, that is, a transformation that, taken $\mathcal{T}$, returns a full infinite tree $\mathcal{T}_W$ having infinite branching degree and embedding $\mathcal{T}$ itself (see Figure 4). This transformation ensures that in $\mathcal{T}_W$ all nodes have the same branching degree and all branches are infinite. To this aim, we use a fresh label # to denote fake nodes, as described in the following definition.

Definition 7.1 (*Widening*). Let $\mathcal{T} = \langle \mathrm{T}, \mathsf{v} \rangle$ be a Σ-labeled $\mathbb{N}$-tree such that $\# \not\in \Sigma$. Then, the *widening* of $\mathcal{T}$ is the Σ_W-labeled $\mathbb{N}$-tree $\mathcal{T}_W \triangleq \langle \mathbb{N}^*, \mathsf{v}_W \rangle$ such that *(i)* $\Sigma_W \triangleq \Sigma \cup \{\#\}$, *(ii)* for $x \in \mathrm{T}$, $\mathsf{v}_W(x) \triangleq \mathsf{v}(x)$, and *(iii)* for $y \in \mathbb{N}^* \setminus \mathrm{T}$, $\mathsf{v}_W(y) \triangleq \#$.

Now, we define a sharp transformation of $\mathcal{T}_W$ in a full binary tree $\mathcal{T}_D$. This is inspired but different from that used to embed the logic $S\omega S$ into $S2S$ [Rabin 1969]. Intuitively, the transformation allows to delay n abstract decisions, to be taken at a node y in $\mathcal{T}_W$ and corresponding to its n successors $y \cdot i$, along some corresponding nodes $x, x \cdot 0, x \cdot 00, \ldots$

in $\mathcal{T}_D$. In particular, when we are on a node $x \cdot 0^i$, we are able to split the decision on $y \cdot i$ into an immediate action, which is sent to the right (effective) successor $x \cdot 0^i \cdot 1$, while the remaining actions are sent to its copy $x \cdot 0^{i+1}$. To differentiate the meaning of left and right successors of a node in $\mathcal{T}_D$, we use the fresh symbol $\perp$ (see Figure 4).

Definition 7.2 (*Delayed Generation*). Let $\mathcal{T}_W = \langle \mathbb{N}^*, \mathsf{v}_W \rangle$ be the widening of a Σ-labeled tree $\mathcal{T}$ such that $\perp \notin \Sigma$. Then, the *delayed generation* of $\mathcal{T}$ is the Σ_D-labeled $\{0,1\}$-tree $\mathcal{T}_D \triangleq \langle \{0,1\}^*, \mathsf{v}_D \rangle$ such that *(i)* $\Sigma_D \triangleq \Sigma_W \cup \{\perp\}$ and *(ii)* there exists a surjective function $\mathsf{s} : \{0,1\}^* \to \mathbb{N}^*$, with $\mathsf{s}(\varepsilon) \triangleq \varepsilon$, $\mathsf{s}(x \cdot 0^i) \triangleq \mathsf{s}(x)$, and $\mathsf{s}(x \cdot 0^i \cdot 1) \triangleq \mathsf{s}(x) \cdot i$, where $x \in \{0,1\}^*$ and $i \in \mathbb{N}$, such that *(ii.i)* $\mathsf{v}_D(x) \triangleq \mathsf{v}_W(\mathsf{s}(x))$, for all $x \in \{\varepsilon\} \cup \{0,1\}^* \cdot \{1\}$, and *(ii.ii)* if $\mathsf{v}_D(x \cdot 1) = \#$ then $\mathsf{v}_D(x \cdot 0) \triangleq \#$ else $\mathsf{v}_D(x \cdot 0) \triangleq \perp$, for all $x \in \{0,1\}^*$.

To complete the tree encoding, we have also to delay the degree associated to each node in the input tree model. We recall that, an original tree model of a graded formula may require a fixed number of paths satisfying the formula going through the same node. Such a number is the degree associated to that node and which we need to delay. To this aim, we enrich the label of a node with a function mapping a set of elements, named *bases*, into triples of numbers representing the splitting of the node degree into two components. The first is the delayed degree, while the second is the degree associated to one of the effective successors of the node. Such a splitting is the delayed abstract action mentioned before, when it is customized to the need of having information on the degrees. We further use a flag with values in $\{\flat, \not\flat\}$ to indicate if the labeling is or not active, that is, if it actually represents the splitting of the degree of a given base that needs to be propagated in the two tree directions. Note that, for a formula with degree g, it is not important to monitor the presence of a finite number of paths of cardinality greater than g. To this purpose, we use the symbol $\not\omega$ to efficiently represent the infinite set $]g, \omega[$. We relate ω and $\not\omega$ to the finite number in $[0, g]$ in the expected way: (i) $i < \not\omega < \omega$, for all $i \in [0, g]$; (ii) $i + j \triangleq \not\omega$, for all $i, j \in [0, g]$ such that $i + j > g$; (iii) $i + j = j + i \triangleq i$, for all $i \in \{\not\omega, \omega\}$ and $j \in [0, g] \cup \{\not\omega, \omega\}$ such that $j \leq i$. The whole idea of the degree encoding is formalized through the following four definitions.

Definition 7.3 ((Σ, B)-*Enriched g-Degree Tree*). Let Σ and B be two sets, $g \in \mathbb{N}$, and $\mathrm{H}(g) \triangleq \{(d, d_1, d_2) \in ([0, g] \cup \{\not\omega, \omega\})^3 : d = d_1 + d_2\} \times \{\flat, \not\flat\}$. Then, a (Σ, B)-*enriched g-degree tree* is a $(\Sigma \times \mathrm{H}(g)^{\mathrm{B}})$-labeled $\{0,1\}$-tree $\mathcal{T} = \langle \{0,1\}^*, \mathsf{v} \rangle$.

We now introduce a (Σ_D, B)-enriched g-degree tree $\mathcal{T}_{D_{\mathrm{B},g}}$ as the extension of the delayed generation $\mathcal{T}_D$ of $\mathcal{T}$ with degree functions in its labeling. Intuitively, each function in a node represents how to distribute and propagate an information on the degrees along its successors.

Definition 7.4 (B-*Based g-Degree Delayed Generation*). Let B be a set, $g \in \mathbb{N}$, and $\mathcal{T}_D = \langle \{0,1\}^*, \mathsf{v}_D \rangle$ be the delayed generation of a Σ-labeled tree $\mathcal{T}$. Then, a B-*based g-degree delayed generation* of $\mathcal{T}$ is a (Σ_D, B)-enriched g-degree tree $\mathcal{T}_{D_{\mathrm{B},g}} = \langle \{0,1\}^*, \mathsf{v}_{D_{\mathrm{B},g}} \rangle$ such that there is an $\mathsf{h} \in \mathrm{H}(g)^{\mathrm{B}}$ with $\mathsf{v}_{D_{\mathrm{B},g}}(x) = (\mathsf{v}_D(x), \mathsf{h})$, for all $x \in \{0,1\}^*$.

In order to have a sound construction for $\mathcal{T}_{D_{\mathrm{B},g}}$, we need to impose a coherence property on the information between a node and its two successors. In particular, whenever we enter a node x labeled with # in its first part, as it represents that the node is fictitious, we have to take no splitting of the degree by sending to x the value 0. In addition, we force children labeled with # to have necessarily the flag set to $\not\flat$. On the other nodes, we need to match the value of the first component of the splitting with the degree of the left successor. Moreover, in dependence of the flag in $\{\flat, \not\flat\}$, we may have also to match the value of the second component with the degree of the right successor. With more details, we require a coherence that is not punctual (=) but rather, depending on the particular kind of bases we are analyzing, it has to be either superior ($\geq$) or inferior ($\leq$) to the value given by the parent of the node. Specifically, to distinguish between these kinds of bases, we split them into the two subsets $\mathrm{B}_{\mathrm{sup}}$ and $\mathrm{B}_{\mathrm{inf}}$. So, a tree has to be superiorly coherent w.r.t. $\mathrm{B}_{\mathrm{sup}}$ and inferiorly coherent w.r.t. $\mathrm{B}_{\mathrm{inf}}$.

Definition 7.5 (GCTL *Sup/Inf Coherence*). Let $\mathcal{T} = \langle \{0,1\}^*, \mathsf{v}\rangle$ be a $(\Sigma \cup \{\#\}, \mathrm{B})$-enriched g-degree tree. Then, $\mathcal{T}$ is *superiorly* (resp., *inferiorly*) *coherent* w.r.t. a base $b \in \mathrm{B}$ if and only if, for $x \in \{0,1\}^*$ and $i \in \{0,1\}$ with $\mathsf{v}(x) = (\sigma, \mathsf{h})$, $\mathsf{h}(b) = (d, d_0, d_1, \beta)$, $\mathsf{v}(x \cdot i) = (\sigma_i, \mathsf{h}_i)$, and $\mathsf{h}_i(b) = (d^i, d^i_0, d^i_1, \beta^i)$, it holds that (i) if $\sigma_i = \#$ then $d_i = 0$ and $\beta^i = \not\flat$ and (ii) if $i = 0$ or $\beta = \flat$ then $d_i \leq d^i$ (resp., $d_i \geq d^i$).

Finally, with the following definition, we extend the local concept of sup/inf coherence of a particular base to a pair of sets of bases $\mathrm{B}_{\mathrm{sup}}, \mathrm{B}_{\mathrm{inf}} \subseteq \mathrm{B}$.

Definition 7.6 (GCTL *Full Coherence*). A $(\Sigma \cup \{\#\}, \mathrm{B})$-enriched g-degree tree $\mathcal{T}$ is *full coherent* w.r.t. a pair $(\mathrm{B}_{\mathrm{sup}}, \mathrm{B}_{\mathrm{inf}})$, where $\mathrm{B}_{\mathrm{sup}} \cup \mathrm{B}_{\mathrm{inf}} \subseteq \mathrm{B}$, if and only if it is superiorly and inferiorly coherent w.r.t. all bases $b \in \mathrm{B}_{\mathrm{sup}}$ and $b \in \mathrm{B}_{\mathrm{inf}}$, respectively.

Note that the sets $\mathrm{B}_{\mathrm{sup}}$ and $\mathrm{B}_{\mathrm{inf}}$ turn out to be useful, in the satisfiability algorithm we give, to deal with the degree of existential and universal path quantifications, respectively. In particular, the whole construction ensures that the degrees of all formulas are correctly propagated along the tree, that is, in other words, that the model is full coherent.

7.2. The Coherence Structure Satellites

We now define the satellites we use to verify that the tree encoding the model of the formula has a correct shape with respect to the whole transformation described in the previous paragraph. In particular, we first introduce a satellite that checks if the "enriched degree tree" in input is the result of a "based degree delayed generation" of the unwinded model of the formula. Then, we show how to create the additional labeling of the tree that satisfies the coherence properties on the degrees required by the semantics of the logic. The following automaton checks if the # and $\perp$ labels of the input tree are correct with respect to Definitions 7.1 and 7.2.

Definition 7.7 (*Structure Satellite*). The *structure satellite* is the satellite $\mathcal{S}^\star \triangleq \langle \Sigma_D, \{0,1\}, \{\#, \perp, @\}, \zeta, \{@\}\rangle$ on binary trees, where ζ is set as follows: if $p = \sigma = \#$ then $\zeta(p, \sigma) \triangleq \{(\#, \#)\}$ else if either $p = \sigma = \perp$ or $p = @$ and $\sigma \in \Sigma$ then $\zeta(p, \sigma) \triangleq \{(\perp, @), (\#, \#)\}$, otherwise $\zeta(p, \sigma) \triangleq \emptyset$.

The satellite $\mathcal{S}^\star$ has constant size 3. Its transition function ζ is defined to directly represent the constraints on the # and $\perp$ labels and, in particular, the state @ is used to represents a real node of the original tree with values in Σ. So, next lemma easily follows.

LEMMA 7.1 (STRUCTURE SATELLITE). *The $\mathcal{S}^\star$ satellite accepts all and only the Σ_D-labeled $\{0,1\}$-trees $\mathcal{T}_D$ that can be obtained as the delayed generation of Σ-labeled $\mathbb{N}$-trees $\mathcal{T}$.*

The next satellite creates the additional labeling of the input tree, for the main automaton, in such a way that it is full coherent w.r.t. the pair of sets $(\mathrm{B}_{\mathrm{sup}}, \mathrm{B}_{\mathrm{inf}})$. Precisely, if the satellite accepts the input tree, the additional labeling of the built tree is given by its states.

Definition 7.8 (GCTL *Coherence Satellite*). The (Σ, B)-*enriched g-degree* $(\mathrm{B}_{\sup}, \mathrm{B}_{\inf})$-*coherence satellite* with $\mathrm{B}_{\sup} \cup \mathrm{B}_{\inf} \subseteq \mathrm{B}$ is the binary satellite $\mathcal{S}_{\mathrm{B},g}^{\Sigma,(\mathrm{B}_{\sup},\mathrm{B}_{\inf})} \triangleq \langle \Sigma \cup \{\#\}, \{0,1\}, \mathrm{H}(g)^{\mathrm{B}}, \zeta, \mathrm{H}(g)^{\mathrm{B}} \rangle$, where ζ is set as follows: (i) if $\sigma = \#$, then $\zeta(p, \sigma) \triangleq \{(p, p)\}$, if for all $b \in \mathrm{B}$ it holds $p(b) = (0, 0, 0, \not\flat)$, and $\zeta(p, \sigma) \triangleq \emptyset$, otherwise; (ii) if $\sigma \neq \#$ then $\zeta(p, \sigma)$ contains all and only the pairs of states $(p_0, p_1) \in (\mathrm{H}(g)^{\mathrm{B}})^{\{0,1\}}$ such that, for all $b \in \mathrm{B}_\alpha$ with $\alpha = \sup$ (resp., $\alpha = \inf$), $p(b) = (d, d_0, d_1, \beta)$, and $p_i(b) = (d^i, d_0^i, d_1^i, \beta^i)$, it holds that if $i = 0$ or $\beta = \flat$ then $d_i \leq d^i$ (resp., $d_i \geq d^i$), for all $i \in \{0, 1\}$.

The transition function is structured to directly represent the constraints of Definitions 7.5 and 7.6. Note that the satellite $\mathcal{S}_{\mathrm{B},g}^{\Sigma,(\mathrm{B}_{\sup},\mathrm{B}_{\inf})}$ is polynomial in g and exponential in $|\mathrm{B}|$, since its number of states is equal to $(2 \cdot (g+3)^2)^{|\mathrm{B}|}$. Next lemma follows by construction.

LEMMA 7.2 (GCTL COHERENCE SATELLITE). *The* $\mathcal{S}_{\mathrm{B},g}^{\Sigma,(\mathrm{B}_{\sup},\mathrm{B}_{\inf})}$ *satellite builds all and only the* $(\Sigma \cup \{\#\}, \mathrm{B})$*-enriched g-degree trees* $\mathcal{T}'$ *over* $\Sigma \cup \{\#\}$*-labeled* $\{0, 1\}$*-tree* $\mathcal{T}$ *that are full coherent w.r.t. the pair* $(\mathrm{B}_{\sup}, \mathrm{B}_{\inf})$*.*

Finally, we introduce the satellite that checks if the tree in input is coherent or not by merging the behavior of the two previous described satellites.

Definition 7.9 (GCTL *Coherence Structure Satellite*). The *B-based g-degree structure* $(\mathrm{B}_{\sup}, \mathrm{B}_{\inf})$-*coherence satellite* with $\mathrm{B}_{\sup} \cup \mathrm{B}_{\inf} \subseteq \mathrm{B}$ is the binary satellite $\mathcal{S}_{\mathrm{B},g}^{\mathrm{B}_{\sup},\mathrm{B}_{\inf}} = \langle \Sigma_D, \{0, 1\}, \mathrm{P}_E \times \mathrm{P}_I, \zeta, \mathrm{P}_{E_0} \times \mathrm{P}_{I_0} \rangle$, where $\mathrm{P}_E = \mathrm{P}_{E_0} \triangleq \mathrm{H}(g)^{\mathrm{B}}$, $\mathrm{P}_I \triangleq \{\#, \bot, @\}$, and $\mathrm{P}_{I_0} \triangleq \{@\}$, obtained as the product of the $(\Sigma \cup \{\bot\}, \mathrm{B})$-enriched g-degree $(\mathrm{B}_{\sup}, \mathrm{B}_{\inf})$-full coherent satellite $\mathcal{S}_{\mathrm{B},g}^{\Sigma\cup\{\bot\},(\mathrm{B}_{\sup},\mathrm{B}_{\inf})}$ with the structure satellite $\mathcal{S}^\star$.

Clearly, the size of $\mathcal{S}_{\mathrm{B},g}^{\mathrm{B}_{\sup},\mathrm{B}_{\inf}}$ is polynomial in g and exponential in $|\mathrm{B}|$, since its number of states is equal to $3 \cdot (2 \cdot (g + 3)^2)^{|\mathrm{B}|}$. Due to the product structure of the automaton, next result directly follows from Lemmas 7.1 and 7.2.

THEOREM 7.1 (GCTL COHERENCE STRUCTURE SATELLITE). *The* $\mathcal{S}_{\mathrm{B},g}^{\mathrm{B}_{\sup},\mathrm{B}_{\inf}}$ *satellite builds all and only the B-based g-degree delayed generations* $\mathcal{T}_{D_{\mathrm{B},g}}$ *of* Σ*-labeled* $\mathbb{N}$*-trees* $\mathcal{T}$ *over their delayed generation* $\mathcal{T}_D$ *that are full coherent w.r.t. the pair* $(\mathrm{B}_{\sup}, \mathrm{B}_{\inf})$*.*

8. GCTL SATISFIABILITY

In this section, we finally introduce an APT $\mathcal{A}_\varphi$ that checks whether a complete 2^{AP}-labeled $\mathbb{N}$-tree $\mathcal{T}$ satisfies a given formula φ by evaluating all B-based g-degree delayed generation trees $\mathcal{T}_{D_{\mathrm{B},g}}$ associated with $\mathcal{T}$, where $g \triangleq \mathring{\varphi}$ is the maximum finite degree of φ and $B \triangleq \mathsf{qcl}(\varphi)$ is the *quantification closure* of φ, that is, the set of all the quantification formulas, contained in the closure, deprived of the degree. To formally define this concept, we first have to introduce the *extended closure* $\mathsf{ecl}(\varphi)$ of a GCTL formula φ that is construct in the same way of $\mathsf{cl}(\varphi)$, by also asserting that (i) if $\mathsf{E}^{\geq g}\varphi_1 \mathsf{Op}\, \varphi_2 \in \mathsf{ecl}(\varphi)$ then $\mathsf{E}^{\geq 1}\varphi_1 \mathsf{Op}\, \varphi_2 \in \mathsf{ecl}(\varphi)$, (ii) if $\mathsf{E}^{\geq g}\varphi_1 \widetilde{\mathsf{Op}}\, \varphi_2 \in \mathsf{ecl}(\varphi)$ then $\mathsf{E}^{\geq 1}\neg(\varphi_1 \widetilde{\mathsf{Op}}\, \varphi_2) \in \mathsf{ecl}(\varphi)$, (iii) if $\mathsf{A}^{<g}\varphi_1 \mathsf{Op}\, \varphi_2 \in \mathsf{ecl}(\varphi)$ then $\mathsf{A}^{<1}\neg(\varphi_1 \mathsf{Op}\, \varphi_2) \in \mathsf{ecl}(\varphi)$, and (iv) if $\mathsf{A}^{<g}\varphi_1 \widetilde{\mathsf{Op}}\, \varphi_2 \in \mathsf{ecl}(\varphi)$ then $\mathsf{A}^{<1}\varphi_1 \widetilde{\mathsf{Op}}\, \varphi_2 \in \mathsf{ecl}(\varphi)$, for all $\mathsf{Op} \in \{\mathsf{U}, \mathsf{R}\}$, and $g \in [2, \omega]$. Intuitively, the difference between $\mathsf{cl}(\varphi)$ and $\mathsf{ecl}(\varphi)$ resides in the fact that, in the latter, we also include the formulas with degree 1 used to deal with the $\equiv_{\mathcal{T}}^{x}$-tautologies and their negations. Note that $|\mathsf{ecl}(\varphi)| = \mathrm{O}(|\mathsf{cl}(\varphi)|)$. The quantification closure is consequently defined as follows: $\mathsf{qcl}_{\mathsf{E}}(\varphi) \triangleq \{\mathsf{E}\psi : \mathsf{E}^{\geq g}\psi \in \mathsf{ecl}(\varphi)\} \setminus \{\mathsf{E}\tilde{\mathsf{X}}\, \mathsf{f}\}$, $\mathsf{qcl}_{\mathsf{A}}(\varphi) \triangleq \{\mathsf{A}\psi : \mathsf{A}^{<g}\psi \in \mathsf{ecl}(\varphi)\} \setminus \{\mathsf{AX}\, \mathsf{t}\}$, and

$\mathsf{qcl}(\varphi) \triangleq \mathsf{qcl}_{\mathsf{E}}(\varphi) \cup \mathsf{qcl}_{\mathsf{A}}(\varphi)$. In particular, observe that we do not need any base for the formulas checking whether there is or not a successor of a node.

The automaton runs on every B-based g-degree generation tree, even those that are not associated with a complete tree. However, we make the assumptions that the trees in input are really associated with this kind of trees and that they are coherent with respect to $(\mathrm{B}_{\mathrm{sup}}, \mathrm{B}_{\mathrm{inf}})$, where $\mathrm{B}_{\mathrm{sup}} \triangleq \mathsf{qcl}_{\mathsf{E}}(\varphi)$ and $\mathrm{B}_{\mathrm{inf}} \triangleq \mathsf{qcl}_{\mathsf{A}}(\varphi)$. By Theorem 7.1, we are able to enforce such properties by using $\mathcal{A}_{\varphi}$ as the main part of an APTS having the B-based g-degree structure $(\mathrm{B}_{\mathrm{sup}}, \mathrm{B}_{\mathrm{inf}})$-coherence satellite $\mathcal{S}_{\mathrm{B},g}^{\mathrm{B}_{\mathrm{sup}},\mathrm{B}_{\mathrm{inf}}}$ as second component.

In order to understand how the formula automaton works, it is useful to gain more insights into the meaning of the tree $\mathcal{T}_{D_{\mathrm{B},g}}$ associated with $\mathcal{T}$. First of all, the widening operation has the purpose to make the tree full by adding fake nodes labeled with #. Through this, we obtain the tree $\mathcal{T}_W$. Then, the delaying operation transforms $\mathcal{T}_W$ into a binary tree $\mathcal{T}_D$, such that at every level a node x associated to a node y in $\mathcal{T}$ generates only one of the successor of y at a time in the direction 1, meanwhile it sends a duplicate of itself on the direction 0 labeled with $\perp$. The following duplicates have to generate the remaining successors in a recursive way. However, if there are no more successors to generate, the node x does not send in the direction 0 a duplicate of itself anymore, but just a fake node labeled with #. At this point, to obtain the tree $\mathcal{T}_{D_{\mathrm{B},g}}$, we enrich the labeling of the delayed generation tree, by adding a degree function $\mathsf{h} : \mathrm{B} \to \mathrm{H}(g)$. In the hypothesis that $\mathcal{T}$ satisfies φ, for every formula $\varphi' \in \mathrm{B}$ and node $x \in \{0, 1\}^*$ with $\mathsf{v}_{D_{\mathrm{B},g}}(x) = (\sigma, \mathsf{h})$, we have that $\mathsf{h}(\varphi') = (d, d_0, d_1, \beta)$ describes the degree with which the formula φ' is supposed to be satisfied on x. In particular d is the degree in the current node, the decomposition $d = d_0 + d_1$ explains how this degree is partitioned in the following left and right children, and the β flag represents whether this splitting of degrees is meaningful or not. More precisely, β is set to $\not\beta$ if and only if the inner formula of φ' or its negation is a structure formula tautology in x. Hence, there is no point in splitting the degree, since the formula is already verified or falsified. Moreover, d_1 represents the degree sent to the direction 1, which usually corresponds to a concrete node in $\mathcal{T}$. Hence, it is the degree sent to that node. Meanwhile, d_0 represents the degree sent to the direction 0, which usually corresponds to a duplicate of the previous node. Hence, d_0 represents the degree that have yet to be partitioned among the remaining successors of the node y associated to x. To this aim, the coherence requirement asks: (i) for an existential formula, the degree found in a successor node is not lower than the degree the father sent to that node (it may be higher as the node may satisfy the formula by finding more paths with a certain property, so it surely satisfies what the formula requires); (ii) for a universal formula, the degree found in a successor node is not greater than the degree the father sent to that node (it may be smaller as the node may satisfy the formula by finding less paths with a certain negated property, so it surely satisfies what the formula requires).

In the hypothesis of coherence, the formula automaton needs only to check that (i) the degree of every existential and universal formula is correctly initiated on the node in which the formula first appears in (e.g., for an existential formula it needs to check that the degree in the label of the node is not lower than the degree required by the formula), and (ii) that every node of the tree satisfies the existential or universal formula with the degree specified in the node labeling. To do this, the automaton $\mathcal{A}_{\varphi}$ has as state space the set $\mathsf{ecl}(\varphi) \cup \mathsf{mcl}(\varphi) \cup \mathsf{qcl}(\varphi) \cup \{\#, \neg\#\}$, where $\mathsf{mcl}(\varphi)$ is the *modified closure* of φ defied as follows: $\mathsf{mcl}(\varphi) \triangleq \mathsf{mcl}_1(\varphi) \cup \mathsf{mcl}_{\omega}(\varphi)$, $\mathsf{mcl}_1(\varphi) \triangleq \mathsf{mcl}_{\mathsf{E}^1}(\varphi) \cup \mathsf{mcl}_{\mathsf{A}^1}(\varphi)$, $\mathsf{mcl}_{\mathsf{E}^1}(\varphi) \triangleq \bigcup_{\mathsf{Op} \in \{\mathsf{U},\mathsf{R}\}}^{i \in \{0,1\}} \mathsf{mcl}_{\mathsf{EOp},i}(\varphi)$, $\mathsf{mcl}_{\mathsf{A}^1}(\varphi) \triangleq \bigcup_{\mathsf{Op} \in \{\mathsf{U},\mathsf{R}\}}^{i \in \{0,1\}} \mathsf{mcl}_{\mathsf{AOp},i}(\varphi)$, $\mathsf{mcl}_{\mathsf{EOp},i}(\varphi) \triangleq \{\mathsf{E}_i^{\geq 1}\psi : \mathsf{E}\psi \in \mathsf{qcl}_{\mathsf{E}}(\varphi) \wedge \psi \in \{\varphi_1 \mathsf{Op}\, \varphi_2, \varphi_1 \widetilde{\mathsf{Op}}\, \varphi_2\}\}$, $\mathsf{mcl}_{\mathsf{AOp},i}(\varphi) \triangleq \{\mathsf{A}_i^{<1}\psi : \mathsf{A}\psi \in \mathsf{qcl}_{\mathsf{A}}(\varphi) \wedge \psi \in \{\varphi_1 \mathsf{Op}\, \varphi_2, \varphi_1 \widetilde{\mathsf{Op}}\, \varphi_2\}\}$, $\mathsf{mcl}_{\omega}(\varphi) \triangleq \mathsf{mcl}_{\mathsf{E}^{\omega}}(\varphi) \cup \mathsf{mcl}_{\mathsf{A}^{\omega}}(\varphi)$, $\mathsf{mcl}_{\mathsf{E}^{\omega}}(\varphi) \triangleq \{\mathsf{E}^{\geq \omega}\psi : \mathsf{E}\psi \in \mathsf{qcl}_{\mathsf{E}}(\varphi)\}$, and $\mathsf{mcl}_{\mathsf{A}^\omega}(\varphi) \triangleq \{\mathsf{A}^{<\omega}\psi : \mathsf{A}\psi \in \mathsf{qcl}_{\mathsf{A}}(\varphi)\}$. On one hand, the formulas in $\mathsf{qcl}(\varphi)$ ask the automaton to verify them completely relying on the degree of the labeling. On the other hand, the existential and universal formulas in $\mathsf{ecl}(\varphi) \cup \mathsf{mcl}(\varphi)$ ask the automaton even to check that their degrees agree with that contained in the labeling. The states # and $\neg$# are used to verify the existence or not of a successor of a node when we have to deal with the formulas $\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f}$ and $\mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t}$. Finally, states in $\mathsf{mcl}(\varphi) \cup \mathsf{qcl}(\varphi)$ are also used for the parity acceptance condition.

Definition 8.1 (GCTL *Formula Automaton*). The formula automaton for φ is the binary APT $\mathcal{A}_\varphi \triangleq \langle \Sigma_\varphi \times \mathrm{P}_{E_\varphi}, \{0, 1\}, \mathrm{Q}_\varphi, \delta, \varphi, \mathrm{F}_\varphi \rangle$, where $\Sigma_\varphi \triangleq 2^{\mathrm{AP}} \cup \{\#, \bot\}$, $\mathrm{P}_{E_\varphi} \triangleq \mathrm{H}(\mathring{\varphi})^{\mathsf{qcl}(\varphi)}$, $\mathrm{Q}_\varphi \triangleq \mathsf{ecl}(\varphi) \cup \mathsf{mcl}(\varphi) \cup \mathsf{qcl}(\varphi) \cup \{\#, \neg\#\}$, $\mathrm{F}_\varphi \triangleq (\mathrm{F}_1, \mathrm{F}_2, \mathrm{Q}_\varphi)$ with $\mathrm{F}_1 \triangleq \mathsf{mcl}_{\mathsf{AU},1}(\varphi) \cup \mathsf{mcl}_{\mathsf{A}^\omega}(\varphi)$ and $\mathrm{F}_2 \triangleq \mathsf{qcl}_{\mathsf{A}}(\varphi) \cup \mathsf{mcl}_{\mathsf{A}^1}(\varphi) \cup \mathsf{mcl}_\omega(\varphi) \cup \mathsf{mcl}_{\mathsf{ER},1}(\varphi)$, and $\delta : \mathrm{Q}_\varphi \times (\Sigma_\varphi \times \mathrm{P}_{E_\varphi}) \to \mathrm{B}^+(\{0, 1\} \times \mathrm{Q}_\varphi)$ is defined in the body of the article.

We now describe the structure of the whole transition function $\delta(q, (\sigma, \mathsf{h}))$ through a case analysis on the state space.

As first thing, when $\sigma = \#$, the automaton is on a fake node $x = x' \cdot i$ of the the input tree $\mathcal{T}_{D_{\mathrm{B},g}}$, so no formula should be checked on it. However, in the instant the automaton reaches such a node, by passing through its antecedent x', it is not asking to verify the formula represented by the state q, due to the fact that it is sent by another state q' on x' which corresponds to a universal formula. In this case, indeed, we are checking that its "core" is satisfied on all successors (but a given number of them). Hence, since x does not exist in the original tree $\mathcal{T}$, we do not have to verify the property of q on it. Moreover, we are sure that q' does not represent any existential property. This is due to the fact that (i) the degree d_i related to the state q' in the labeling of x' needs to be 0 by the coherence requirements of Definition 7.5 and (ii), as we show later, the transition on existential formulas do not send any state to a direction $j \in \{0, 1\}$ having $d_j = 0$. For this reason, we set $\delta(q, (\#, \mathsf{h})) \triangleq \mathfrak{t}$, for all $q \in \mathrm{Q}_\varphi$ and $\mathsf{h} \in \mathrm{P}_{E_\varphi}$.

Furthermore, the structure of the transition function does not send a state q belonging to the set $(\mathsf{ecl}(\varphi) \setminus \mathsf{mcl}_\omega(\varphi)) \cup \bigcup_{\mathsf{Op}\in\{\mathsf{U},\mathsf{R}\}}(\mathsf{mcl}_{\mathsf{EOp},1}(\varphi) \cup \mathsf{mcl}_{\mathsf{AOp},1}(\varphi))$ to a node labeled with $\sigma = \bot$ and a state q belonging to the set $\bigcup_{\mathsf{Op}\in\{\mathsf{U},\mathsf{R}\}}(\mathsf{mcl}_{\mathsf{EOp},0}(\varphi) \cup \mathsf{mcl}_{\mathsf{AOp},0}(\varphi))$ to a node labeled with $\sigma \neq \bot$. For this reason, w.l.o.g., we can set $\delta(q, (\sigma, \mathsf{h})) \triangleq \mathfrak{f}$, for all these cases.

Now, we describe the remaining part of the definition of $\delta(q, (\sigma, \mathsf{h}))$ with the proviso that (i) $\sigma \neq \#$, (ii) if $q \in (\mathsf{ecl}(\varphi) \setminus \mathsf{mcl}_\omega(\varphi)) \cup \bigcup_{\mathsf{Op}\in\{\mathsf{U},\mathsf{R}\}}(\mathsf{mcl}_{\mathsf{EOp},1}(\varphi) \cup \mathsf{mcl}_{\mathsf{AOp},1}(\varphi))$ then $\sigma \neq \bot$, and (iii) if $q \in \bigcup_{\mathsf{Op}\in\{\mathsf{U},\mathsf{R}\}}(\mathsf{mcl}_{\mathsf{EOp},0}(\varphi) \cup \mathsf{mcl}_{\mathsf{AOp},0}(\varphi))$ then $\sigma = \bot$.

(1) If $q \in \mathrm{Lit} \triangleq \mathrm{AP} \cup \neg\mathrm{AP}$, where $\neg\mathrm{AP} \triangleq \{\neg p : p \in \mathrm{AP}\}$, the automaton has to verify if the literal is locally satisfied or not. To do this, we set $\delta(q, (\sigma, \mathsf{h})) \triangleq \mathfrak{t}$, if either $q \in \mathrm{AP}$ and $q \in \sigma$ or $q \in \neg\mathrm{AP}$ and $q \notin \sigma$, and $\delta(q, (\sigma, \mathsf{h})) \triangleq \mathfrak{f}$, otherwise.
(2) The Boolean cases are treated in the classical way: $\delta(\varphi_1 \wedge \varphi_2, (\sigma, \mathsf{h})) \triangleq \delta(\varphi_1, (\sigma, \mathsf{h})) \wedge \delta(\varphi_2, (\sigma, \mathsf{h}))$ and $\delta(\varphi_1 \vee \varphi_2, (\sigma, \mathsf{h})) \triangleq \delta(\varphi_1, (\sigma, \mathsf{h})) \vee \delta(\varphi_2, (\sigma, \mathsf{h}))$.
(3) The case $\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f}$ (resp., $\mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t}$) is simply solved by setting $\delta(\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\mathfrak{f}, (\sigma, \mathsf{h})) \triangleq (1, \#)$ (resp., $\delta(\mathsf{A}^{<1}\mathsf{X}\,\mathfrak{t}, (\sigma, \mathsf{h})) \triangleq (1, \neg\#)$) and $\delta(\#, (\sigma, \mathsf{h})) \triangleq \mathfrak{t}$ (resp., $\delta(\neg\#, (\sigma, \mathsf{h})) \triangleq \mathfrak{f}$), if $\sigma = \#$, and $\delta(\#, (\sigma, \mathsf{h})) \triangleq \mathfrak{f}$ (resp., $\delta(\neg\#, (\sigma, \mathsf{h})) \triangleq \mathfrak{t}$), otherwise.
(4) Let $\mathsf{h}(\mathsf{EX}\,\varphi) = (d, d_0, d_1, \beta)$ (resp., $\mathsf{h}(\mathsf{A}\tilde{\mathsf{X}}\,\varphi) = (d, d_0, d_1, \beta)$). For a state of the form $\mathsf{EX}\,\varphi$ (resp., $\mathsf{A}\tilde{\mathsf{X}}\,\varphi$) we verify that this formula holds with degree d. The flag β needs to be $\not{b}$, since a next formula on a successor node is not related to one in the current node, due to the fact that this kind of formula never propagate itself. Recall that in the input tree the pair of degrees (d_0, d_1) describe the distribution of the degree d on

the nodes, which need to (resp., are allowed to not) satisfy φ, among the successors of the current node. Since the nodes on the direction 1 are real successors of the node in the original input tree $\mathcal{T}$ we need to ask that the state formula φ holds on them if and only if $d_1 = 1$ (resp., $d_1 = 0$). However, we cannot ask that a state formula holds more than one time, so, if $d_1 > 1$, the input tree cannot be accepted, since $\mathsf{E}^{\geq d_1}\varphi \equiv \mathfrak{f}$ (resp., we do not make any difference in dependence of a value $d_1 > 0$, since $\mathsf{A}^{\leq d_1}\varphi \equiv \mathfrak{t}$). Finally, on direction 0, we send the same state $\mathsf{EX}\,\varphi$ (resp., $\mathsf{A}\tilde{\mathsf{X}}\,\varphi$) if $0 < d_0 < \omega$ (resp., $0 \leq d_0 < \not\omega$), in order to ask that the residual degree d_0 is distributed on the remaining successors. When we deal with the infinite degree ω (resp., finite but unbounded degree $\not\omega$) we have to ensure that the formula φ is verified infinitely often (resp. falsified finitely often) on the successors of the current node. To this aim, every time a nonnull degree is sent to direction 1, we sent the state $\mathsf{E}^{\geq\omega}\mathsf{X}\,\varphi$ (resp. $\mathsf{A}^{<\omega}\mathsf{X}\,\varphi$) to direction 0. Formally, $\delta(\mathsf{EX}\,\varphi, (\sigma, h))$ (resp., $\delta(\mathsf{A}\tilde{\mathsf{X}}\,\varphi, (\sigma, h))$) is set to $\mathfrak{f}$, if $\beta = \flat$, and to the following conjunction, otherwise:

—$\begin{cases} \mathfrak{t}, & \text{if } d_0 = 0; \\ (0, \mathsf{EX}\,\varphi), & \text{if } d_0 < \omega; \\ (0, \mathsf{EX}\,\varphi), & \text{if } d_0 = \omega \text{ and } d_1 = 0; \\ (0, \mathsf{E}^{\geq\omega}\mathsf{X}\,\varphi), & \text{if } d_0 = \omega \text{ and } d_1 \neq 0; \end{cases} \wedge \begin{cases} \mathfrak{t}, & \text{if } d_1 = 0; \\ (1, \varphi), & \text{if } d_1 = 1; \\ \mathfrak{f}, & \text{if } d_1 > 1. \end{cases}$

—$\begin{cases} (0, \mathsf{A}\tilde{\mathsf{X}}\,\varphi), & \text{if } d_0 < \not\omega; \\ (0, \mathsf{A}\tilde{\mathsf{X}}\,\varphi), & \text{if } d_0 = \not\omega \text{ and } d_1 \neq 0; \\ (0, \mathsf{A}^{<\omega}\tilde{\mathsf{X}}\,\varphi), & \text{if } d_0 = \not\omega \text{ and } d_1 = 0; \\ \mathfrak{f}, & \text{if } d_0 = \omega; \end{cases} \wedge \begin{cases} (1, \varphi), & \text{if } d_1 = 0; \\ \mathfrak{t}, & \text{if } d_1 > 0. \end{cases}$

For a state of the form $\mathsf{E}^{\geq g}\mathsf{X}\,\varphi$ (resp., $\mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi$) we have only to further verify that the degree g agrees with the value d, that is, $d \geq g$ (resp., $d < g$). Formally, $\delta(\mathsf{E}^{\geq g}\mathsf{X}\varphi, (\sigma, \mathsf{h}))$ (resp., $\delta(\mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi, (\sigma, \mathsf{h}))$) is set to $\mathfrak{f}$, if $d < g$ (resp., $d \geq g$), and to $\delta(\mathsf{EX}\,\varphi, (\sigma, h))$ (resp., $\delta(\mathsf{A}\tilde{\mathsf{X}}\,\varphi, (\sigma, h))$), otherwise.

(5) A state $\mathsf{E}_i^{\geq 1}\psi$ (resp., $\mathsf{A}_i^{<1}\psi$) in $\mathsf{mcl}(\varphi)$ is used to verify that there is a branch satisfying (resp., all branch satisfy) the inner path formula $\psi = \varphi_1 \mathsf{Op} \varphi_2$, regardless the precise value of the added degree labels. What is important is only to follow paths in which the degrees are not null (for the existential case only). The related transition function simply reflects the one-step unfolding of the CTL formulas, shown in Proposition 3.3. When this requirement needs to be propagated on some successor node, we send different states in the two tree directions, with the sole purpose to distinguish these ones for acceptance reasons.

—$\delta(\mathsf{E}_i^{\geq 1}\varphi_1 \mathsf{U}\, \varphi_2, (\sigma, \mathsf{h})) \triangleq \delta(\varphi_2, (\sigma, \mathsf{h})) \vee \delta(\varphi_1, (\sigma, \mathsf{h})) \wedge \bigvee_{j \in \{0,1\}}^{d_j > 0} (j, \mathsf{E}_j^{\geq 1}\varphi_1 \mathsf{U}\, \varphi_2)$;

—$\delta(\mathsf{A}_i^{<1}\varphi_1 \mathsf{U}\, \varphi_2, (\sigma, \mathsf{h})) \triangleq \delta(\varphi_2, (\sigma, \mathsf{h})) \vee \delta(\varphi_1, (\sigma, \mathsf{h})) \wedge \bigwedge_{j \in \{0,1\}} (j, \mathsf{A}_j^{<1}\varphi_1 \mathsf{U}\, \varphi_2) \wedge \delta(\mathsf{A}^{<1}\mathsf{X}\, \mathfrak{t}, (\sigma, \mathsf{h}))$;

—$\delta(\mathsf{E}_i^{\geq 1}\varphi_1 \mathsf{R}\, \varphi_2, (\sigma, \mathsf{h})) \triangleq \delta(\varphi_2, (\sigma, \mathsf{h})) \wedge (\delta(\varphi_1, (\sigma, \mathsf{h})) \vee \bigvee_{j \in \{0,1\}}^{d_j > 0} (j, \mathsf{E}_j^{\geq 1}\varphi_1 \mathsf{R}\, \varphi_2))$;

—$\delta(\mathsf{A}_i^{<1}\varphi_1 \mathsf{R}\, \varphi_2, (\sigma, \mathsf{h})) \triangleq \delta(\varphi_2, (\sigma, \mathsf{h})) \wedge (\delta(\varphi_1, (\sigma, \mathsf{h})) \vee \bigwedge_{j \in \{0,1\}} (j, \mathsf{A}_j^{<1}\varphi_1 \mathsf{R}\, \varphi_2) \wedge \delta(\mathsf{A}^{<1}\mathsf{X}\, \mathfrak{t}, (\sigma, \mathsf{h})))$;

—$\delta(\mathsf{E}_i^{\geq 1}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2, (\sigma, \mathsf{h})) \triangleq \delta(\varphi_2, (\sigma, \mathsf{h})) \vee \delta(\varphi_1, (\sigma, \mathsf{h})) \wedge (\bigvee_{j \in \{0,1\}}^{d_j > 0} (j, \mathsf{E}_j^{\geq 1}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2) \vee \delta(\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\, \mathfrak{f}, (\sigma, \mathsf{h})))$;

—$\delta(\mathsf{A}_i^{<1}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2, (\sigma, \mathsf{h})) \triangleq \delta(\varphi_2, (\sigma, \mathsf{h})) \vee \delta(\varphi_1, (\sigma, \mathsf{h})) \wedge \bigwedge_{j \in \{0,1\}} (j, \mathsf{A}_j^{<1}\varphi_1 \tilde{\mathsf{U}}\, \varphi_2)$;

—$\delta(\mathsf{E}_i^{\geq 1}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2, (\sigma, \mathsf{h})) \triangleq \delta(\varphi_2, (\sigma, \mathsf{h})) \wedge (\delta(\varphi_1, (\sigma, \mathsf{h})) \vee \bigvee_{j \in \{0,1\}}^{d_j > 0} (j, \mathsf{E}_j^{\geq 1}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2) \vee \delta(\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\, \mathfrak{f}, (\sigma, \mathsf{h})))$;

—$\delta(\mathsf{A}_i^{<1}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2, (\sigma, \mathsf{h})) \triangleq \delta(\varphi_2, (\sigma, \mathsf{h})) \wedge (\delta(\varphi_1, (\sigma, \mathsf{h})) \vee \bigwedge_{j \in \{0,1\}} (j, \mathsf{A}_j^{<1}\varphi_1 \tilde{\mathsf{R}}\, \varphi_2))$.

For a state of the form $\mathsf{E}^{\geq 1}\psi$ (resp., $\mathsf{A}^{<1}\psi$) we have only to further verify that $d \geq 1$ (resp., $d < 1$). Formally, $\delta(\mathsf{E}^{\geq 1}\psi, (\sigma, \mathsf{h}))$ (resp., $\delta(\mathsf{A}^{<1}\psi, (\sigma, \mathsf{h}))$) is set to $\mathfrak{f}$, if $d < 1$ (resp., $d \geq 1$), and to $\delta(\mathsf{E}^{\geq 1}_1\psi, (\sigma, h))$ (resp., $\delta(\mathsf{A}^{<1}_1\psi, (\sigma, h))$), otherwise.

(6) Let $\mathsf{h}(\mathsf{E}\psi) = (d, d_0, d_1, \beta)$ (resp., $\mathsf{h}(\mathsf{A}\psi) = (d, d_0, d_1, \beta)$), where $\psi = \varphi_1 \mathsf{Op}\, \varphi_2$. For a state of the form $\mathsf{E}\psi$ (resp., $\mathsf{A}\psi$) we verify that this formula holds with degree d. If the node is not a duplicate of a previous node, that is, $\sigma \neq \bot$, we have to check the formula, which should hold in the current node, by applying the one-step unfolding property derived by the semantics and reported in Corollary 5.2. At this point, we may need to propagate the formula in the two directions of the tree, by taking into account the requirements established by the degree in those directions. If such degree d_i is 0 (resp., ω) then the existential (resp., universal) formula is immediately true (resp., false). If $d_i = 1$ (resp., $d_i = 0$), we propagate a particular requirement with the meaning that we are looking for a path (resp., all paths) satisfying the internal path formula $\varphi_1 \mathsf{Op}\, \varphi_2$. Precisely, in order to differentiate between the two directions, we send the state $\mathsf{E}^{\geq 1}_i \varphi_1 \mathsf{Op}\, \varphi_2$ (resp., $\mathsf{A}^{<1}_i \varphi_1 \mathsf{Op}\, \varphi_2$) to direction $i \in \{0, 1\}$. If $d_i > 1$ (resp., $0 < d_i < \omega$) we propagate the original requirement by leaving to the degree of the successor nodes the task to specify how many paths (resp., do not) satisfy the inner formula. However, when we deal with the infinite degree ω (resp., finite but unbounded degree $\not\omega$) we have to ensure that the formula $\varphi_1 \mathsf{Op}\, \varphi_2$ is verified on infinitely (resp. falsified on finitely) many paths. To this aim, we use the apposite state $\mathsf{E}^{\geq \omega} \varphi_1 \mathsf{Op}\, \varphi_2$ (resp., $\mathsf{A}^{<\omega} \varphi_1 \mathsf{Op}\, \varphi_2$), which is sent on one direction if and only if on the other one there is a nonnull (resp., null) degree. In this way, we can keep track of a possible infinite splitting of the degree which is required (resp., forbidden) by an infinite (resp., finite) number of paths. In the following we describe such a propagation of the states by means of the following macro: $\gamma_{\mathsf{EOp}}(d_0, d_1) \triangleq \gamma^0_{\mathsf{EOp}}(d_0, d_1) \wedge \gamma^1_{\mathsf{EOp}}(d_0, d_1)$ (resp., $\gamma_{\mathsf{AOp}}(d_0, d_1) \triangleq \gamma^0_{\mathsf{AOp}}(d_0, d_1) \wedge \gamma^1_{\mathsf{AOp}}(d_0, d_1)$), where

$$—\gamma^i_{\mathsf{EOp}}(d_0, d_1) \triangleq \begin{cases} \mathfrak{t}, & \text{if } d_i = 0; \\ (i, \mathsf{E}^{\geq 1}_i \varphi_1 \mathsf{Op}\, \varphi_2), & \text{if } d_i = 1; \\ (i, \mathsf{E}\varphi_1 \mathsf{Op}\, \varphi_2), & \text{if } d_i < \omega; \\ (i, \mathsf{E}\varphi_1 \mathsf{Op}\, \varphi_2), & \text{if } d_i = \omega \text{ and } d_{1-i} = 0; \\ (i, \mathsf{E}^{\geq \omega} \varphi_1 \mathsf{Op}\, \varphi_2) \wedge & \\ \wedge\, (1-i, \mathsf{E}^{\geq 1}_{1-i} \varphi_1 \mathsf{Op}\, \varphi_2), & \text{if } d_i = \omega \text{ and } d_{1-i} \neq 0. \end{cases}$$

$$—\gamma^i_{\mathsf{AOp}}(d_0, d_1) \triangleq \begin{cases} (i, \mathsf{A}^{<1}_i \varphi_1 \mathsf{Op}\, \varphi_2), & \text{if } d_i = 0; \\ (i, \mathsf{A}\varphi_1 \mathsf{Op}\, \varphi_2), & \text{if } d_i < \not\omega; \\ (i, \mathsf{A}\varphi_1 \mathsf{Op}\, \varphi_2), & \text{if } d_i = \not\omega \text{ and } d_{1-i} = 0; \\ (i, \mathsf{A}^{<\omega} \varphi_1 \mathsf{Op}\, \varphi_2), & \text{if } d_i = \not\omega \text{ and } d_{1-i} \neq 0; \\ \mathfrak{f}, & \text{if } d_i = \omega. \end{cases}$$

Observe that the last case requires the existence of a path satisfying the inner formula ψ in the direction $1 - i$. This is due to the fact that, when we verify the existential formula with infinite degree, we risk that the latter is always regenerated without actually completing a real path satisfying ψ. By coupling this condition with that about the infinite generation, we ensure that we actually find infinitely many paths satisfying ψ. (Resp., the first to last case may also require that in the direction $1 - i$ there is no path falsifying the inner formula ψ. However, this requirement is implicit in the whole structure of $\gamma_{\mathsf{AOp}}(d_0, d_1)$.)

(7) Let $\mathsf{h}(\mathsf{Qn}\, \psi) = (d, d_0, d_1, \beta)$ with $\psi = \varphi_1 \mathsf{Op}\, \varphi_2$. Due to the meaning of the flag β, when $\beta = \not\flat$, the automaton has to verify that either ψ or $\neg\psi$ is a tautology. On the contrary, when $\beta = \flat$, it has to verify that no one of them is a tautology. Thus,

we need two components of the transition function, $\eta_\psi(\sigma, \mathsf{h})$ and $\overline{\eta}_\psi(\sigma, \mathsf{h})$, to ensure, respectively, that ψ is or not a tautology on a node labeled with σ. These components have to require the automaton to check the truth of the formulas equivalent to the tautologies, as described in Theorem 5.2.

—$\eta_{\varphi_1 \mathsf{U} \varphi_2}(\sigma, \mathsf{h}) \triangleq \delta(\varphi_2, (\sigma, \mathsf{h}))$;
—$\overline{\eta}_{\varphi_1 \mathsf{U} \varphi_2}(\sigma, \mathsf{h}) \triangleq \delta(\neg\varphi_2, (\sigma, \mathsf{h}))$;
—$\eta_{\varphi_1 \mathsf{R} \varphi_2}(\sigma, \mathsf{h}) \triangleq \delta(\varphi_1, (\sigma, \mathsf{h})) \wedge \delta(\varphi_2, (\sigma, \mathsf{h}))$;
—$\overline{\eta}_{\varphi_1 \mathsf{R} \varphi_2}(\sigma, \mathsf{h}) \triangleq \delta(\neg\varphi_1, (\sigma, \mathsf{h})) \vee \delta(\neg\varphi_2, (\sigma, \mathsf{h}))$;
—$\eta_{\varphi_1 \tilde{\mathsf{U}} \varphi_2}(\sigma, \mathsf{h}) \triangleq \delta(\mathsf{A}^{<1}\varphi_1 \tilde{\mathsf{U}} \varphi_2, (\sigma, \mathsf{h}))$;
—$\overline{\eta}_{\varphi_1 \tilde{\mathsf{U}} \varphi_2}(\sigma, \mathsf{h}) \triangleq \delta(\mathsf{E}^{\geq 1}\neg(\varphi_1 \tilde{\mathsf{U}} \varphi_2), (\sigma, \mathsf{h}))$;
—$\eta_{\varphi_1 \tilde{\mathsf{R}} \varphi_2}(\sigma, \mathsf{h}) \triangleq \delta(\mathsf{A}^{<1}\varphi_1 \tilde{\mathsf{R}} \varphi_2, (\sigma, \mathsf{h}))$;
—$\overline{\eta}_{\varphi_1 \tilde{\mathsf{R}} \varphi_2}(\sigma, \mathsf{h}) \triangleq \delta(\mathsf{E}^{\geq 1}\neg(\varphi_1 \tilde{\mathsf{R}} \varphi_2), (\sigma, \mathsf{h}))$.

(8) Now, we discuss the general structure of a transition function for a state of the form $\mathsf{E}\psi$ (resp., $\mathsf{A}\psi$) with $\psi = \varphi_1 \mathsf{Op}\, \varphi_2$. Let $\mathsf{h}(\mathsf{E}\psi) = (d, d_0, d_1, \beta)$ (resp., $\mathsf{h}(\mathsf{A}\psi) = (d, d_0, d_1, \beta)$). Note that the degree d is never equal to 0 or 1 (resp. 0 or ω), because the requirement $\gamma_{\mathsf{EOp}}(d_0, d_1)$ (resp., $\gamma_{\mathsf{AOp}}(d_0, d_1)$) discussed before never propagates an existential (resp., universal) state without degree on a direction i when $d_i = 0$ or $d_i = 1$ (resp. $d_i = 0$ or $d_i = \omega$). If the node is not a duplicate of a previous node, that is, $\sigma \neq \bot$, we verify that the formula holds in the current node by applying the one-step unfolding property derived by the semantics, as reported in Corollary 5.2. Precisely, since $d > 1$ (resp. $0 < d < \omega$) ψ cannot (resp., can) be a tautology, otherwise (resp., since) we would find only one minimal path satisfying ψ. On the other hand, $\neg\psi$ cannot (resp., can) be a tautology, otherwise (resp., since) we would find only one minimal path non satisfying ψ. So, the automaton has to verify that ψ and $\neg\psi$ are not tautologies in the current node and has to propagate the existential state on the successors through the $\gamma_{\mathsf{EOp}}(d_0, d_1)$ requirement (resp., the automaton has to verify either that ψ or $\neg\psi$ is a tautology or that both are not tautologies and that the universal requirement $\gamma_{\mathsf{AOp}}(d_0, d_1)$ is propagated on the successors). Due to the nontautological nature of ψ and $\neg\psi$, the automaton has to reject the input tree when $\beta = \not\flat$ (resp. the automaton has to verify that ψ or $\neg\psi$ is a tautology if and only if $\beta = \not\flat$). If $\sigma = \bot$ then the current node is simply a replica of a previous node with $\sigma \neq \bot$. Since the existential (resp., universal) state have been propagated on direction 0, we already know that ψ and $\neg\psi$ are not tautologies, hence we need just to propagate the state through the relative $\gamma_{\mathsf{EOp}}(d_0, d_1)$ (resp., $\gamma_{\mathsf{AOp}}(d_0, d_1)$) requirement. Therefore, the, When $\sigma = \bot$, both ψ and $\neg\psi$ are not tautologies; automaton has to reject the tree when $\beta = \not\flat$.

$$—\delta(\mathsf{E}\psi, (\sigma, \mathsf{h})) \triangleq \begin{cases} \mathfrak{f}, & \text{if } \beta = \not\flat; \\ \gamma_{\mathsf{EOp}}(d_0, d_1), & \text{if } \sigma = \bot \text{ and } \beta = \flat; \\ \overline{\eta}_\psi(\sigma, \mathsf{h}) \wedge \overline{\eta}_{\neg\psi}(\sigma, \mathsf{h}) \wedge \gamma_{\mathsf{EOp}}(d_0, d_1), & \text{if } \sigma \neq \bot \text{ and } \beta = \flat. \end{cases}$$

$$—\delta(\mathsf{A}\psi, (\sigma, \mathsf{h})) \triangleq \begin{cases} \mathfrak{f}, & \text{if } \sigma = \bot \text{ and } \beta = \not\flat; \\ \gamma_{\mathsf{AOp}}(d_0, d_1), & \text{if } \sigma = \bot \text{ and } \beta = \flat; \\ \eta_\psi(\sigma, \mathsf{h}) \vee \eta_{\neg\psi}(\sigma, \mathsf{h}), & \text{if } \sigma \neq \bot \text{ and } \beta = \not\flat; \\ \overline{\eta}_\psi(\sigma, \mathsf{h}) \wedge \overline{\eta}_{\neg\psi}(\sigma, \mathsf{h}) \wedge \gamma_{\mathsf{AOp}}(d_0, d_1), & \text{if } \sigma \neq \bot \text{ and } \beta = \flat. \end{cases}$$

Note that, the whole transition function can be simplified, case by case, because of the redundancy of some of its components. For example, consider the case EU when $\sigma \neq \bot$ and $\beta = \flat$. By definition, we obtain that $\delta(\mathsf{E}\varphi_1 \mathsf{U}\, \varphi_2, (\sigma, \mathsf{h})) = \delta(\neg\varphi_2, (\sigma, \mathsf{h})) \wedge \delta(\mathsf{E}^{\geq 1}\varphi_1 \mathsf{U}\, \varphi_2, (\sigma, \mathsf{h})) \wedge \gamma_{\mathsf{EU}}(d_0, d_1)$, which can be equivalently written as follows: $\delta(\neg\varphi_2, (\sigma, \mathsf{h})) \wedge \delta(\varphi_1, (\sigma, \mathsf{h})) \wedge \delta(\mathsf{E}^{\geq 1}\mathsf{X}\, \mathsf{E}^{\geq 1}\varphi_1 \mathsf{U}\, \varphi_2, (\sigma, \mathsf{h})) \wedge \gamma_{\mathsf{EU}}(d_0, d_1)$. Now, since the requirement $\gamma_{\mathsf{EU}}(d_0, d_1)$ ensure the existence of $d = d_0 + d_1 > 1$ non equivalent paths starting on the successors of the current node, we have that the $\delta(\mathsf{E}^{\geq 1}\mathsf{X}\,\mathsf{E}^{\geq 1}\varphi_1 \mathsf{U}\,\varphi_2, (\sigma, \mathsf{h}))$ component is surely verified. So, this piece is redundant. The remaining expression $\delta(\neg\varphi_2, (\sigma, \mathsf{h})) \wedge \delta(\varphi_1, (\sigma, \mathsf{h})) \wedge \gamma_{\mathsf{EU}}(d_0, d_1)$ simply reflects what is required by Item 5.2 of Corollary 5.2. Now, for a state of the form $\mathsf{E}^{\geq g}\psi$ (resp., $\mathsf{A}^{<g}\psi$), with $g \in [2, \omega]$, we have only to further verify that $d \geq g$ (resp., $d < g$). Formally, $\delta(\mathsf{E}^{\geq g}\psi, (\sigma, \mathsf{h}))$ (resp., $\delta(\mathsf{A}^{<g}\psi, (\sigma, \mathsf{h}))$) is set to $\mathfrak{f}$, if $d < g$ (resp., $d \geq g$), and to $\delta(\mathsf{E}\psi, (\sigma, h))$ (resp., $\delta(\mathsf{A}\psi, (\sigma, h))$), otherwise.

We now briefly discuss the parity acceptance condition for $\mathcal{A}_\varphi$. Note that, in our reasonings, we assume $\mathrm{F}_\varphi = (\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3)$ with $\mathrm{F}_3 = \mathrm{Q}_\varphi$.

Let $\mathcal{T}$ be a complete $\mathbb{N}$-tree, $\mathcal{T}_{D_{\mathrm{B},g}}$ be one of its B-based g-degree delayed generation in input to $\mathcal{A}_\varphi$, and $\mathcal{R}$ be a related run. It is easy to see that states in $\mathsf{cl}(\varphi) \setminus \mathsf{mcl}_\omega(\varphi)$ represents literals, ands, ors, and quantified formulas with finite degree that never generate themselves, so, they never progress infinitely often. On the other hand, formulas in $\mathsf{mcl}(\varphi) \cup \mathsf{qcl}(\varphi)$ may be generated infinitely often, but only some of them should be allowed to do so (due to their intrinsic semantics).

(1) Existential next states $\mathsf{EX}\,\varphi$ and $\mathsf{E}^{\geq\omega}\mathsf{X}\,\varphi$ are never sent to direction 1 and they can only progress indefinitely along direction 0. The propagation of an existential formula without degree represents a delay of the choice of the particular successors of the replicated node on which it is needed to verify φ. When the associated degree is finite, the formula needs to be satisfied on a finite number of successors. So, the choice of the successors must be eventually made, and the formula cannot be propagated indefinitely. When the degree is infinite, instead, the formula is allowed to progress under the condition that successors satisfying ψ are found infinitely often. Hence, we use two states: a ω-grade version is generated every time a successor satisfying φ is found and a gradeless version is used when the successor is skipped. Hence, the existential next formulas $\mathsf{EX}\,\varphi$ is not allowed to progress indefinitely and, thus, it belongs to F_3 but not to F_2. On the other hand the formulas $\mathsf{E}^{\geq\omega}\mathsf{X}\,\varphi$ are allowed to occur infinitely often and, thus, they belong to F_2 but not F_1.
(2) Universal next states $\mathsf{AX}\,\varphi$ and $\mathsf{A}^{\geq\omega}\mathsf{X}\,\varphi$ are never sent to direction 1 and they can only progress indefinitely along direction 0. An infinite generation of an universal next formulas represents the propagation of a requirement demanded on infinitely many successors of the replicated node with the aim to check that only a finite number of them do not satisfy it. This should be allowed, however, when the associated degree is finite but not a priori determined, that is, if it is $\not\omega$. Generally, this degree can be split infinitely many times without decreasing, so, we risk to allow infinitely many successors to not satisfy φ. In order to avoid such a problem, we use two states: a ω-grade version is generated every time a successor is allowed to not satisfy $\neg\varphi$ and a gradeless version is used when the successor satisfies φ. Hence, the universal formulas $\mathsf{AX}\,\varphi$ is allowed to progress indefinitely on such branches and, thus, it belongs to F_2 but not to F_1. On the other hand the universal formula $\mathsf{A}^{<\omega}\mathsf{X}\,\varphi$ is not allowed to occur infinitely often, even when $\mathsf{AX}\,\varphi$ does, thus, it belongs to F_1.
(3) Existential nonnext formulas $\mathsf{E}_i^{\geq 1}\psi$, with degree 1, have to trace a path satisfying the inner path formula $\psi \in \{\varphi_1 \mathsf{U}\,\varphi_2, \varphi_1 \mathsf{R}\,\varphi_2, \varphi_1 \tilde{\mathsf{U}}\,\varphi_2, \varphi_1 \tilde{\mathsf{R}}\,\varphi_2\}$. When ψ is an until or weak until formula, the path have to eventually reach a point in which the formula is locally satisfied. So, the relative states $\mathsf{E}_i^{\geq 1}\psi$ are not allowed to progress indefinitely and, thus, they belong to F_3 but not to F_2. When ψ is a release or weak release formula, it may happened that there are no points in which the formula is

locally satisfied. However, only paths that progress infinitely often along direction 1 are real paths of the input tree (following the replica indefinitely would yield no path). Hence, states $\mathsf{E}_0^{\geq 1}\psi$ belong to F_3 but not to F_2, and states $\mathsf{E}_1^{\geq 1}\psi$ belong to F_2 but not F_1.

(4) Universal nonnext formulas $\mathsf{A}_i^{<1}\psi$, with degree 1, have to trace all paths and prove that they satisfy the inner path formula $\psi \in \{\varphi_1 \mathsf{U}\, \varphi_2, \varphi_1 \mathsf{R}\, \varphi_2, \varphi_1 \tilde{\mathsf{U}}\, \varphi_2, \varphi_1 \tilde{\mathsf{R}}\, \varphi_2\}$. When ψ is a release or weak release formula, it may happened that there are no points in which the formula is locally satisfied. So, the relative states $\mathsf{A}_i^{<1}\psi$ are allowed to progress indefinitely and, thus, they belong to F_2 but not to F_1. When ψ is an until or weak until formula, the path have to eventually reach a point in which the formula is locally satisfied. However, we need to propagate it infinitely often along direction 0, in order to ask it on all successor of the replicated node. Now, since on paths that progress infinitely often along direction 1 it is possible to generate both the states $\mathsf{A}_0^{\geq 1}\psi$ and $\mathsf{A}_1^{\geq 1}\psi$, the infinite generation of $\mathsf{A}_1^{\geq 1}\psi$ has an higher nonacceptance priority with respect to that of $\mathsf{A}_0^{\geq 1}\psi$. This is due to the fact that those paths represent real branches of the input tree where ψ need to eventually hold. Hence, states $\mathsf{A}_0^{\geq 1}\psi$ belong to F_2 but not to F_1, and states $\mathsf{A}_1^{\geq 1}\psi$ belong to F_1.

(5) Existential nonnext formulas with infinite degree $\mathsf{E}^{\geq\omega}\psi$ or without degree $\mathsf{E}\psi$ have to trace a non singleton set of paths satisfying the inner path formula $\psi \in \{\varphi_1 \mathsf{U}\, \varphi_2, \varphi_1 \mathsf{R}\, \varphi_2, \varphi_1 \tilde{\mathsf{U}}\, \varphi_2, \varphi_1 \tilde{\mathsf{R}}\, \varphi_2\}$. One one hand, if the number of such paths is finite, the automaton will eventually reach a node from which there in only one outgoing path model of ψ, since all the paths have to eventually split. When this happens, the automaton verify the existence of such a path with the relative 1-grade version $\mathsf{E}_i^{\geq 1}\psi$. Hence, when a gradeless formula is accompanied by a finite degree it must not progress infinitely often. On the other hand, when the number of paths the automaton needs to follow is infinite, we should allow the existential formula to progress infinitely often. However, by doing so, we risk to trace just one path in the input tree along which we propagate the existential formula and, obviously, it cannot provide the infinite number of paths we need in order to verify the formula itself. Thus, when we propagate the existential requirement on direction i, we have to use the two versions of the requirement itself. The ω-grade formula is sent on direction i when on direction $1-i$ is ensured the existence of a path satisfying ψ. Instead, the gradeless version is used when such an existence is not verified. Consequently, when the ω-grade version is generated infinitely often along the path, there are infinite branches coming out from this and satisfying ψ. On the contrary, when the gradeless version is definitively propagated, we are just following a unique path which cannot provide the infinite paths we need. Hence, all gradeless nonnext existential formula belong to F_3 but not to F_2 and their ω-grade versions belong to F_2 but not to F_1.

(6) Universal nonnext formulas with infinite degree $\mathsf{A}^{<\omega}\psi$ or without degree $\mathsf{A}\psi$ have to trace a set of paths that are allowed to not satisfy the inner path formula $\psi \in \{\varphi_1 \mathsf{U}\, \varphi_2, \varphi_1 \mathsf{R}\, \varphi_2, \varphi_1 \tilde{\mathsf{U}}\, \varphi_2, \varphi_1 \tilde{\mathsf{R}}\, \varphi_2\}$. There may be cases in which the automaton eventually reach a node from which there are no outgoing paths model of $\neg\psi$. When this happens, the automaton needs to verify the universal validity of ψ with the relative 1-grade version $\mathsf{A}_i^{<1}\psi$. Also, the automaton may reach a point where the ψ or $\neg\psi$ are tautologies and, thus, it stops by verifying one of them. However, it is also possible that the universal requirement progress infinitely often. In such a case, we have that it is tracing one path that may not satisfy ψ, even if it would be allowed to trace more paths. Since the accompanying degree is greater than 0, this does not result to be a problem and, hence, we allow the infinite propagation. Moreover, every

time we meet an universal formula with finite but non a priori determined degree, that is, if such degree is $\not\omega$, the formula may split in the two direction and allow paths to not satisfy the ψ formula on both of them. If this happens infinitely often along the single path on which we are propagating the requirement, we would allow an infinite numbers of path to not satisfy ψ, contradicting what we want to verify. Thus, when we propagate the universal requirement on direction i, we have to use the two versions of the requirement itself. The ω-grade formula is sent on direction i when on direction $1-i$ is allowed the existence of a path nonsatisfying ψ. Instead, the gradeless version is used when such an existence is forbidden. Consequently, when the ω-grade version is generated infinitely often along the path, there may be infinite branches coming out from this and nonsatisfying ψ. On the contrary, when the gradeless version is definitively propagated, we are just following a unique path which does not allow the existence of the infinite number of paths we want to avoid. Hence, all gradeless nonnext universal formula belong to F_2 but not to F_1 and their ω-grade versions belong to F_1.

We now prove the following main result about the decidability of GCTL satisfiability.

THEOREM 8.1 (GCTL SATISFIABILITY). *Let φ be a* GCTL *formula, with $g = \mathring{\varphi}$,* $\mathrm{B} = \mathsf{qcl}(\varphi)$*,* $\mathrm{B}_{\sup} = \mathsf{qcl}_{\mathsf{E}}(\varphi)$*, and* $\mathrm{B}_{\inf} = \mathsf{qcl}_{\mathsf{A}}(\varphi)$*. Then, φ is satisfiable if and only if* $\mathrm{L}(\langle \mathcal{A}_\varphi, \mathcal{S}_{\mathrm{B},g}^{\mathrm{B}_{\sup},\mathrm{B}_{\inf}} \rangle) \neq \emptyset$.

PROOF. [Only if]. Given a 2^{AP}-labeled $\mathbb{N}$-tree $\mathcal{T} = \langle \mathrm{T}, \mathsf{v} \rangle$ model of φ, we first show how to recursively construct one of its B-based g-degree delayed generation trees $\mathcal{T}_{D_{\mathrm{B},g}} = \langle \{0,1\}^*, \mathsf{v}_{D_{\mathrm{B},g}} \rangle$, necessarily full coherent w.r.t. the pair $(\mathrm{B}_{\sup}, \mathrm{B}_{\inf})$, along with a partial map $\mathsf{t} : \{0,1\}^* \rightharpoonup \mathrm{T}$ that links each node $x \in \{0,1\}^*$ of $\mathcal{T}_{D_{\mathrm{B},g}}$, with $\mathsf{v}_{D_{\mathrm{B},g}}(x) = (\sigma, \mathsf{h})$ and $\sigma \neq \#$, to the corresponding one $\mathsf{t}(x) \in \mathrm{T}$ in $\mathcal{T}$. This function, is simply the restriction to real nodes, that is, nodes not labeled with #, of the s function introduced in Definition 7.2 of the delayed generation.

To each subtree $\mathcal{T}_{D_{\mathrm{B},g}}^{x}$ of $\mathcal{T}_{D_{\mathrm{B},g}}$ rooted in $x = x' \cdot 0^j$, with $x' \in \{\varepsilon\} \cup 0^* \cdot 1$, $\mathsf{v}_{D_{\mathrm{B},g}}(x) = (\sigma, \mathsf{h})$ such that $\sigma \neq \#$, we associate the subtree $\mathcal{T}^x$ of $\mathcal{T}$ rooted in $y = \mathsf{t}(x)$. Observe that $\mathcal{T}^{x \cdot 1}$ is the subtree of $\mathcal{T}$ rooted at the $(j+1)$-th successor of y and that $\mathcal{T}^{x \cdot 0} = \mathcal{T}^x$. Moreover, by $\mathcal{T}'^x$ we denote the subtree of $\mathcal{T}^x$ in which the first j successors of the root are deleted. Note that $\mathcal{T}'^{x \cdot 1} = \mathcal{T}^{x \cdot 1}$ and $\mathcal{T}'^{x \cdot 0}$ is the subtree of $\mathcal{T}'^x$ with the first successor of the root deleted.

In the rest of the proof, we say that a path formula ψ is *locally determined* on a node x if and only if either ψ or $\neg\psi$ is an $\equiv_{\mathcal{T}^x}^{\varepsilon}$-tautology.

For each node $x \in \{0,1\}^*$ and base $b \in \mathrm{B}$ with $\mathsf{v}_{D_{\mathrm{B},g}}(x) = (\sigma, \mathsf{h})$, $\mathsf{h}(b) = (d, d_0, d_1, \beta)$, $\mathsf{v}_{D_{\mathrm{B},g}}(x \cdot 0) = (\sigma_0, \mathsf{h}_0)$, and $\mathsf{v}_{D_{\mathrm{B},g}}(x \cdot 1) = (\sigma_1, \mathsf{h}_1)$ we set: if $\sigma = \#$ then $d = d_0 = d_1 \triangleq 0$ and $\beta \triangleq \not\flat$, if $\sigma_0 = \#$ then $d_0 \triangleq 0$, if $\sigma_1 = \#$ then $d_1 \triangleq 0$. For the other cases, we set the values of the degrees as follows, where we recall that $\not\omega$ is in place of any finite number greater than g.

(1) $b = \mathsf{EX}\,\varphi$. Then, $\beta \triangleq \not\flat$ and d (resp., d_0) is set to the maximum degree $l \in [0, g] \cup \{\not\omega, \omega\}$ with which the formula $\mathsf{E}^{\geq l}\mathsf{X}\,\varphi$ is satisfied on $\mathcal{T}'^x$ (resp., $\mathcal{T}'^{x \cdot 0}$ if $\sigma_0 \neq \#$). Moreover, d_1 is set to 1, if φ is satisfied on $\mathcal{T}'^{x \cdot 1}$, and to 0 otherwise.
(2) $b = \mathsf{A}\tilde{\mathsf{X}}\,\varphi$. Then, $\beta \triangleq \not\flat$ and d (resp., d_0) is set to the minimum degree $l \in [0, g] \cup \{\not\omega, \omega\}$ with which the formula $\mathsf{A}^{<l+1}\tilde{\mathsf{X}}\,\varphi$ is satisfied on $\mathcal{T}'^x$ (resp., $\mathcal{T}'^{x \cdot 0}$ if $\sigma_0 \neq \#$). Moreover, d_1 is set to 1, if φ is not satisfied on $\mathcal{T}'^{x \cdot 1}$, and to 0 otherwise.
(3) $b = \mathsf{E}\psi$ is a nonnext formula. Then, $\beta \triangleq \not\flat$ if ψ is locally determined on x. If $\beta = \flat$, then d (resp., d_0, d_1) is set to the maximum degree $l \in [0, g] \cup \{\not\omega, \omega\}$ with which the

formula $\mathsf{E}^{\geq l}\mathsf{X}\,\psi$ (resp., $\mathsf{E}^{\geq l}\mathsf{X}\,\psi$, $\mathsf{E}^{\geq l}\psi$) is satisfied on $\mathcal{T}'^{x}$ (resp., $\mathcal{T}'^{x\cdot 0}$ if $\sigma_0 \neq \#$, $\mathcal{T}'^{x\cdot 1}$ if $\sigma_1 \neq \#$). If $\beta = \not\flat$, only d and d_0 are set as stated before, while d_1 is arbitrary.

(4) $b = \mathsf{A}\psi$ is a nonnext formula. Then, $\beta \triangleq \not\flat$ if ψ is locally determined on x. If $\beta = \flat$, then d (resp., d_0, d_1) is set to the minimum degree $l \in [0, g] \cup \{\not{o}, \omega\}$ with which the formula $\mathsf{A}^{<l+1}\mathsf{X}\,\psi$ (resp., $\mathsf{A}^{<l+1}\mathsf{X}\,\psi$, $\mathsf{A}^{<l+1}\psi$) is satisfied on $\mathcal{T}'^{x}$ (resp., $\mathcal{T}'^{x\cdot 0}$ if $\sigma_0 \neq \#$, $\mathcal{T}'^{x\cdot 1}$ if $\sigma_1 \neq \#$). If $\beta = \not\flat$, only d and d_0 are set as stated before, while d_1 is arbitrary.

It is immediate to see that, if $\beta = \flat$ then $d = d_0 + d_1$. Moreover, let $\mathsf{h}_0(b) = (d^0, d_0^0, d_1^0, \beta^0)$ and $\mathsf{h}_1(b) = (d^1, d_0^1, d_1^1, \beta^1)$, we have that $d^0 = d_0$ and if $\beta = \flat$ then $d^1 = d_1$. Now, by Definition 7.6, we can derive that the tree $\mathcal{T}_{D_{\mathrm{B},g}}$ is actually full coherent w.r.t. the pair $(\mathrm{B}_{\sup}, \mathrm{B}_{\inf})$. Hence, by Theorem 7.1, we have that it can be obtained as a building of the satellite $\mathcal{S}_{\mathrm{B},g}^{\mathrm{B}_{\sup},\mathrm{B}_{\inf}}$ over the delayed generation $\mathcal{T}_D$ of $\mathcal{T}$ itself.

It remains to prove that $\mathcal{T}_{D_{\mathrm{B},g}}$ is accepted by $\mathcal{A}_\varphi$. The proof proceeds by induction on the structure of the set of states derived by the formula φ and on the degree d associated to the state. In particular, we use the following ordering $\prec \subseteq \mathsf{Q} \times \mathsf{Q}$ between states: (i) for all formulas $\varphi', \varphi'' \in \mathsf{Q}$ with $\varphi'' \in \mathsf{ecl}(\varphi')$ and $\varphi'' \neq \varphi'$, we set $\varphi'' \prec \varphi'$; (ii) $\mathsf{E}\psi \prec \mathsf{E}^{\geq l}\psi$ (resp., $\mathsf{A}\psi \prec \mathsf{A}^{<l}\psi$) and $\mathsf{E}^{\geq\omega}\psi \prec \mathsf{E}^{\geq l}\psi$ (resp., $\mathsf{A}^{<\omega}\psi \prec \mathsf{A}^{<l}\psi$), for all $l \in [2, \omega[$; (iii) $\mathsf{E}^{\geq 1}\psi \prec \mathsf{E}\psi$ (resp., $\mathsf{A}^{<1}\psi \prec \mathsf{A}\psi$) and $\mathsf{E}^{\geq 1}\psi \prec \mathsf{E}^{\geq\omega}\psi$ (resp., $\mathsf{A}^{<1}\psi \prec \mathsf{A}^{<\omega}\psi$); (iv) $\mathsf{E}_i^{\geq 1}\psi \prec \mathsf{E}^{\geq 1}\psi$ (resp., $\mathsf{A}_i^{<1}\psi \prec \mathsf{A}^{<1}\psi$), for all $i \in \{0, 1\}$. We also use the following inductive hypotheses: (i) each state $q = \mathsf{E}\psi$ is sent to a node x with the related degree greater than 1, that is, with $\mathsf{v}_{D_{\mathrm{B},g}}(x) = (\sigma, \mathsf{h})$, $\mathsf{h}(q) = (d, d_0, d_1, \beta)$, and $d > 1$; (ii) each state $q = \mathsf{E}^{\geq\omega}\psi$ is sent to a node x with infinite related degree, that is, with $\mathsf{v}_{D_{\mathrm{B},g}}(x) = (\sigma, \mathsf{h})$, $\mathsf{h}(\mathsf{E}\psi) = (d, d_0, d_1, \beta)$, and $d = \omega$.

Intuitively, if the automaton $\mathcal{A}_\varphi$ is on a state $q = \mathsf{Qn}\;\psi$ (resp. $q = \mathsf{Qn}\,\mathsf{X}\,\psi$), where Qn is a quantification, on a node x of the tree $\mathcal{T}_{D_{\mathrm{B},g}}$, with label $\mathsf{v}_{D_{\mathrm{B},g}}(x) = (\sigma, \mathsf{h})$ and $\sigma \neq \#$, then it accepts the subtree $\mathcal{T}_{D_{\mathrm{B},g}}^{x}$ if either it is able to check the truth of formulas of lower order than q w.r.t. $\prec$, implying already the validity of q itself, or it checks other formulas lower than ψ w.r.t. $\prec$, implying the nonvalidity of the negation of the formula represented by q, and verifies that the subtree $\mathcal{T}'^{x}$ satisfies the formula represented by $\mathsf{Qn}\,\mathsf{X}\,\psi$ (resp., $\mathsf{Qn}\;\psi$) with degree given either by the formula q itself or, if such degree is not present in it, by the d component of the function h valuated on the relative base.

We now give a detailed explanation only for the inductive case of $q = \mathsf{E}\psi$ with $\psi = \varphi_1 \mathsf{Op}\, \varphi_2$, when we are on a node $x = x' \cdot 1$. The other cases are a variation on theme.

Let $\mathsf{h}(q) = (d, d_0, d_1, \beta)$. By the inductive hypothesis, the degree d is greater than 1. Hence ψ is not a tautology (otherwise, we would find only one path satisfying ψ). So, we have $\beta = \flat$. Consequently, the related path formulas $\mathsf{X}\,\psi$ and ψ are true on some of the successors of $\mathsf{t}(x)$ partitioned between $\mathcal{T}'^{x\cdot 0}$ and $\mathcal{T}'^{x\cdot 1}$. Precisely, we have $\mathsf{E}^{=d_0}\mathsf{X}\,\psi$ is satisfied on $\mathcal{T}'^{x\cdot 0}$ and $\mathsf{E}^{=d_1}\psi$ is satisfied on $\mathcal{T}^{x\cdot 1}$. The transition function checks that ψ and $\neg\psi$ are not tautologies, by verifying formulas of lower order than ψ w.r.t. $\prec$, through the use of the components $\overline{\eta}_\psi(\sigma, \mathsf{h})$ and $\overline{\eta}_{\neg\psi}(\sigma, \mathsf{h})$. Moreover, the transition function verifies the same state q on $\mathcal{T}'^{x\cdot 0}$ and $\mathcal{T}^{x\cdot 1}$, through the component $\gamma_{\mathsf{EOp}}(d_0, d_1)$. Observe that this formula sends the states $\mathsf{E}\psi$ and $\mathsf{E}^{\geq\omega}\psi$ on direction i only if $d_i > 1$ and $d_i = \omega$, respectively.

At this point, we have to distinguish between the two cases $d < \omega$ and $d = \omega$.

In the first, it is possible that the automaton needs to check only states of lower order w.r.t. $\prec$, so the acceptance is deduced by the inductive hypothesis. On the contrary, it may also happen that the state propagates itself with the same degree on one direction. But, this propagation cannot happen indefinitely, since the degree eventually splits, and so, it eventually incurs in the first possibility.

In the second case, instead, the state q surely propagates on one direction q itself or its ω-degree version $\mathsf{E}^{\geq\omega}\psi$. So, the induction does not reach a lower case. Let $t = x_0 \cdot x_1 \cdots$ with $x_0 = x$ be the branch on which the infinite degree d is propagated: formally, for each $k \in \mathbb{N}$ with $\mathsf{v}_{D_{\mathrm{B},g}}(x_k) = (\sigma, \mathsf{h})$ and $\mathsf{h}(\mathsf{E}\psi) = (d^k, d_0^k, d_1^k, \beta^k)$, we have $d^k = \omega$. Moreover, let $\mathsf{f} : \mathbb{N} \to \{0, 1\}$ be the direction function that associates to each index $k \in \mathbb{N}$ the direction of the successor of x_k, that is, $x_{k+1} = x_k \cdot \mathsf{f}(k)$. Then, we distinguish the two following cases, where only the first one can actually happen, meanwhile the second one yield a contradiction.

(1) $d^k_{1-\mathsf{f}(k)} > 0$, for infinitely many $k \in \mathbb{N}$. In this case, the automaton passes, on the branch t, through the state $\mathsf{E}^{\geq\omega}\psi$ infinitely often, so it accepts the branch t.
(2) $d^k_{1-\mathsf{f}(k)} = 0$, for all $k \in \mathbb{N}$. We distinguish two sub-cases: t progresses definitively on the direction 0 and t progresses infinitely often through direction 1.
   (a) $\mathsf{f}(k) = 0$ so, $x_k = x_0 \cdot 0^k$, for all $k \in \mathbb{N}$. By construction of $\mathcal{T}^{x_0}$, we have that the tree $\mathcal{T}^{x_k \cdot 1}$ does not contain a path that satisfies the formula, for all $k \in \mathbb{N}$. This means that there is no path satisfying the formula through any successor of $\mathsf{t}(x_0)$. But this contradicts the hypothesis that $\mathcal{T}^x$ satisfies the q with infinite degree.
   (b) $\mathsf{f}(k) = 1$, for infinitely many $k \in \mathbb{N}$. Than, there is an infinite set of indexes $\{j_0, j_1, \ldots\} \subseteq \mathbb{N}$ with $j_0 = 0$ such that, for all $l \in \mathbb{N}$ and $k \in [j_l, j_{l+1}[$, it holds that $x_k = x_{j_l} \cdot 0^{k-j_l}$, and $x_{j_{l+1}} = x_{j_l-1} \cdot 1$. Let $y_l = \mathsf{t}(x_{j_l})$, for all $l \in \mathbb{N}$. Then, the branch $r = y_0 \cdot y_1 \cdots$ is an infinite path in $\mathcal{T}^{x_0}$ on which there are infinite nonequivalent paths that starting in y_l and satisfying ψ, for all $l \in \mathbb{N}$. Now, since $d^k_{1-\mathsf{f}(k)} = 0$, all these paths have to pass through y_{l+1}. By induction, we obtain that all the paths that start from y_0 and satisfy ψ must pass through all the nodes of r. But this is a contradiction, since it means that they are actually one unique path.

[If]. The converse direction is specular. Since a tree $\mathcal{T}_D$ is accepted by $\langle \mathcal{A}_\varphi, \mathcal{S}^{\mathrm{B_{sup}},\mathrm{B_{inf}}}_{\mathrm{B},g} \rangle$, we can assert that *(i)* it is actually a delayed generation of a 2^{AP}-labeled tree $\mathcal{T}$ and *(ii)* the B-based g-degree delayed generation tree $\mathcal{T}_{D_{\mathrm{B},g}}$ built by the satellite $\mathcal{S}^{\mathrm{B_{sup}},\mathrm{B_{inf}}}_{\mathrm{B},g}$ on $\mathcal{T}_D$ is full coherent w.r.t. $(\mathrm{B_{sup}}, \mathrm{B_{inf}})$ and it is accepted by $\mathcal{A}_\varphi$. Using these facts, by induction on the structure of the formula, we can prove that every time $\mathcal{A}_\varphi$ is in a state q on a node x of the tree $\mathcal{T}_{D_{\mathrm{B},g}}$ with label (σ, h), $\mathcal{T}^x$ satisfies the formula represented by q with the related degree if and only if the automaton accepts the subtree $\mathcal{T}^x_{D_{\mathrm{B},g}}$. Actually, this fact happens if x is a right node, that is, when x does not terminates with 0. When x is a left node, the transition function only requires that $\mathcal{T}^x$ satisfies the next formulas in the one-step unfolding of q. However, since the formulas not in the scope of the next are yet verified on a previous right node, we also obtain that $\mathcal{T}^x$ satisfies the whole q. Finally, since $\mathcal{A}_\varphi$ accepts $\mathcal{T}^\varepsilon_{D_{\mathrm{B},g}}$ by hypothesis, we have that the tree $\mathcal{T}$ is a model of φ. □

By a matter of calculation, it holds that $|\mathcal{A}_\varphi| = \mathrm{O}(|\varphi|)$ and $|\mathcal{S}^{\mathrm{B_{sup}},\mathrm{B_{inf}}}_{\mathrm{B},g}| = 2^{\mathrm{O}(|\varphi| \cdot \log(\mathring{\varphi}))}$. Moreover, also the alphabet $\Sigma_\varphi \times \mathrm{P}_{E_\varphi}$ of the APTS has size $2^{\mathrm{O}(|\varphi| \cdot \log(\mathring{\varphi}))}$. By Theorem 6.1, we obtain that the emptiness problem for $\langle \mathcal{A}_\varphi, \mathcal{S}^{\mathrm{B_{sup}},\mathrm{B_{inf}}}_{\mathrm{B},g} \rangle$ can be solved in time $2^{\mathrm{O}(|\varphi|^2 \cdot (\log(|\varphi|) + \log(\mathring{\varphi})))} \leq 2^{\mathrm{O}(\|\varphi\|^3)}$. Moreover, by recalling that GCTL subsumes CTL, the following result follows.

THEOREM 8.2 (GCTL SATISFIABILITY COMPLEXITY). *The satisfiability problem for* GCTL *with binary coding of degrees is* EXPTIME-COMPLETE.

9. CONCLUSION

Graded modalities refine classical existential and universal quantifiers by specifying the number of elements for which the existential requirement should hold/universal requirement may not hold. Earlier work studied the extension of the μCALCULUS by graded modality on successors and shown that the complexity of the related satisfiability problem stays EXPTIME-COMPLETE. In this paper, we have introduced GCTL as an extension of CTL with graded modalities on paths, in order to count the number of equivalence classes of paths satisfying a given formula. We have proposed a general framework that allows to define different kinds of "graded extensions" of GCTL, depending on the specific equivalence relation one chooses among paths. Moreover, we have described reasonable properties that such an equivalence should satisfy and, as a concrete application of our general framework, we have studied a graded logic with path prefix equivalence based on the suitable concepts of minimality and conservativeness. This choice is aimed on counting a minimal way a Kripke structure has to satisfy a given formula in such a way we can ensure its satisfiability no matter how a minimal part is extended.

One of the main features of GCTL is the capability to express properties that are weaker than those definable with the universal quantifications $\mathsf{A}\psi$ and stronger than those definable with the existential quantifications $\mathsf{E}\psi$. In *"planning in nondeterministic domain"* [Cimatti et al. 1998, 2003], for example, the use of strong planning (that is, all the goals have to be satisfied by all the computations) and weak planning (i.e., all the goals have to be satisfied by some computation) are two extreme ways to achieve a given purpose. With our logic, we are able to express "graded path specification" that can be considered as a compromise between strong and weak planning.

We have studied several properties of GCTL under the path prefix equivalence and all of them hold in the general case of graded numbers coded in binary. Among the others, we have proved that this logic can be reduced to the GμCALCULUS, but that it is at least exponentially more succinct. Also, we have studied the satisfiability problem and, by using a sharp automata-theoretic approach via a binary-tree encoding of models and a refinement of the technique involving satellite automata, we have shown that this problem is EXPTIME-COMPLETE, thus no harder than the one for CTL. This result, along with the fact that GCTL is exponentially more succinct of GμCALCULUS and much more "friendly" to use, make GCTL a very useful and powerful logic to be used in practice in formal system verification. It is important to note that, all the results we have achieved for GCTL with path prefix equivalence are based on the properties we have studied for a general path equivalence. Hence, all the technical constructions can be easily lifted to any other graded extension of CTL that respects those properties.

As we have reported before, our satisfiability algorithm for GCTL uses an automata-theoretic approach on the binary-tree encoding of the models of the formula. While the automata approach results in a natural and classical one, it may be also substituted by other techniques, such as the systems of infinite tableaux [Friedmann et al. 2010], turning in an algorithm with the same overall complexity we achieve. On the contrary, the binary-tree encoding seems to be unavoidable even in the case of the tableaux approach. Indeed, by using the regular models, we need to label each node with a tuple of degree functions, used for the splitting, which are not of fixed size 3 anymore, but rather linear in the degree of the formula and so exponential in its size. Then, by applying either the automata or the tableaux approach it turns in an overall double-exponential algorithm.

As future work, there are several directions that could be investigated along with graded path modalities. In particular, we left open the solution of the satisfiability problem of GCTL*. However, by a simple variation of the technique developed in this paper, one can easily obtain a 3EXPTIME upper bound, while a 2EXPTIME-HARD lower bound easily derives from the satisfiability of CTL*. In this case, is also worth investigating the use of the tableaux technique to try to match the known lower bound. By exploiting a similar idea of that used for GμCALCULUS, one could also investigate whether GCTL* is equivalent to CTL* augmented with graded world modalities (Counting-CTL* [Moller and Rabinovich 2003]). However, we conjecture that GCTL* is exponentially more succinct than Counting-CTL* (for GCTL and Counting-CTL, this result holds by simply applying the same idea used for the translation from GCTL to the GμCALCULUS). This result is important as it was shown in Moller and Rabinovich [2003] that Counting-CTL* is equivalent to monadic path logic, which is MSOL with set quantifications restricted to paths.

APPENDIX

A. MATHEMATICAL NOTATION

Classic Objects. Given two *sets* X and Y of *objects*, we denote by $|X|$ the *cardinality* of X, that is, the number of its elements, by 2^X the *powerset* of X, that is, the set of all its subsets, and by $Y^X \subseteq 2^{X \times Y}$ the set of *total functions* f from the *domain* $\mathsf{dom}(\mathsf{f}) \triangleq X$ to the *codomain* $\mathsf{cod}(\mathsf{f}) \triangleq Y$. In addition, with $\mathsf{rng}(\mathsf{f}) \triangleq \{\mathsf{f}(x) : x \in X\} \subseteq \mathsf{cod}(\mathsf{f})$ we indicate the *range* of f, that is, the set of values actually assumed by f. Often, we write $\mathsf{f} : X \to Y$ and $\mathsf{f} : X \rightharpoonup Y$ to indicate, respectively, $\mathsf{f} \in Y^X$ and $\mathsf{f} \in \bigcup_{X' \subseteq X} Y^{X'}$. Regarding the latter, note that we consider f as a *partial function* from X to Y, where $\mathsf{dom}(\mathsf{f}) \subseteq X$ contains all and only the elements for which f is defined. Given a set Z, by $\mathsf{f}_{\restriction Z} \triangleq \mathsf{f} \cap (Z \times Y)$ we denote the *restriction* of f to the set $X \cap Z$, that is, the function $\mathsf{f}_{\restriction Z} : X \cap Z \rightharpoonup Y$ such that, for all $x \in \mathsf{dom}(\mathsf{f}) \cap Z$, it holds that $\mathsf{f}_{\restriction Z}(x) = \mathsf{f}(x)$. Moreover, with $\varnothing$ we indicate a generic *empty function*, that is, a function with empty domain. Note that $X \cap Z = \emptyset$ implies $\mathsf{f}_{\restriction Z} = \varnothing$.

As special sets, we consider $\mathbb{N}$ as the set of *natural numbers* and $[m, n] \triangleq \{k \in \mathbb{N} : m \leq k \leq n\}$, $[m, n[\triangleq \{k \in \mathbb{N} : m \leq k < n\}$, $]m, n] \triangleq \{k \in \mathbb{N} : m < k \leq n\}$, and $]m, n[\triangleq \{k \in \mathbb{N} : m < k < n\}$ as its *interval* subsets, with $m \in \mathbb{N}$ and $n \in \widehat{\mathbb{N}} \triangleq \mathbb{N} \cup \{\omega\}$, where ω is the *numerable infinity*, that is, the *least infinite ordinal*.

Words. By X^n with $n \in \mathbb{N}$ we denote the set of all *n-tuples* of elements from X, by $X^* \triangleq \bigcup_{n=0}^{<\omega} X^n$ the set of *finite words* on the *alphabet* X, by $X^+ \triangleq X^* \setminus \{\varepsilon\}$ the set of *nonempty words*, and by X^ω the set of *infinite words*, where, as usual, $\varepsilon \in X^*$ is the *empty word*. Moreover, $|x| \in \widehat{\mathbb{N}}$ indicates the *length* of a word $x \in X^\infty \triangleq X^* \cup X^\omega$. By $(x)_i$ we denote the *i-th letter* of the finite word x, with $i \in [0, |x|[$. Furthermore, by $\mathsf{fst}(x) \triangleq (x)_0$ (resp., $\mathsf{lst}(x) \triangleq (x)_{|x|-1}$), we indicate the *first* (resp., *last*) letter of x. In addition, by $x_{\leq i}$ (resp., $x_{>i}$), we denote the *prefix* up to (resp., *suffix* after) the letter of index i of x, that is, the finite word built by the first $i + 1$ (resp., last $|x| - i - 1$) letters $(x)_0, \ldots, (x)_i$ (resp., $(x)_{i+1}, \ldots, (x)_{|x|-1}$). We also set, $x_{<0} \triangleq \varepsilon$, $x_{<i} \triangleq x_{\leq i-1}$, $x_{\geq 0} \triangleq x$, and $x_{\geq i} \triangleq x_{>i-1}$, for $i \in [1, |x|[$. Mutatis mutandis, the notations of i-th letter, first, prefix, and suffix apply to infinite words too. Finally, by $\mathsf{pfx}(x_1, x_2) \in X^\infty$ we indicate the *maximal common prefix* of two different words $x_1, x_2 \in X^\infty$, that is, the finite word $x \in X^*$ for which there are two words $x'_1, x'_2 \in X^\infty$ such that $x_1 = x \cdot x'_1$, $x_2 = x \cdot x'_2$, and $\mathsf{fst}(x'_1) \neq \mathsf{fst}(x'_2)$. By convention, we set $\mathsf{pfx}(x, x) \triangleq x$.

Trees. For a set Δ of objects, called *directions*, a Δ*-tree* is a set $T \subseteq \Delta^*$ closed under prefix, that is, if $t \cdot d \in T$, with $d \in \Delta$, then also $t \in T$, and we say that it is *complete* if and only if it also holds that $t \cdot d' \in T$, for all $d' < d$, where $< \subseteq \Delta \times \Delta$ is a fixed strict total order on the directions that is clear from the context. The elements of T are called *nodes* and the empty word ε is the *root* of T. For every $t \in T$ and $d \in \Delta$, the node $t \cdot d \in T$ is a *successor* of t in T. T is *full* if and only if $T = \Delta^*$. Moreover, it is *b-bounded* if and only if the maximal number b of its node successors is finite, that is, $b = \max_{t \in \mathrm{T}} |\{t \cdot d \in \mathrm{T} : d \in \Delta\}| < \infty$. A *branch* of a tree T is a subset $\mathrm{T}' \subseteq \mathrm{T}$ closed under prefix such that, for each $t \in \mathrm{T}'$, there exists at most one successor $t \cdot d \in \mathrm{T}'$. For a finite set Σ of objects, called *symbols*, a Σ-*labeled* Δ-*tree* is a pair $\langle \mathrm{T}, \mathsf{v} \rangle$, where T is a Δ-tree and $\mathsf{v} : \mathrm{T} \rightarrow \Sigma$ is a *labeling function*. When Δ and Σ are clear from the context, we call $\langle \mathrm{T}, \mathsf{v} \rangle$ simply a (labeled) tree.

B. LTL SEMANTICS

In this short appendix, we report the definition of the semantics of all LTL formulas ψ w.r.t. finite and infinite words $\varpi \in (2^{\mathrm{AP}})^{\infty}$, with $\varpi \neq \varepsilon$, on the alphabet 2^{AP}.

(1) $\varpi \models p$, for $p \in \mathrm{AP}$, if and only if $p \in \varpi_0$.
(2) $\varpi \models \neg\psi$ if and only if not $\varpi \models \psi$, that is $\varpi \not\models \psi$;
(3) $\varpi \models \psi_1 \wedge \psi_2$ if and only if $\varpi \models \psi_1$ and $\varpi \models \psi_2$;
(4) $\varpi \models \psi_1 \vee \psi_2$ if and only if $\varpi \models \psi_1$ or $\varpi \models \psi_2$;
(5) $\varpi \models \mathsf{X}\, \psi$ if and only if $|\varpi| > 1$ and $\varpi_{\geq 1} \models \psi$;
(6) $\varpi \models \psi_1 \mathsf{U}\, \psi_2$ if and only if there is an index $i \in [0, |\varpi|[$ such that $\varpi_{\geq i} \models \psi_2$ and, for all indexes $j \in [0, i[$, it holds that $\varpi_{\geq j} \models \psi_1$;
(7) $\varpi \models \psi_1 \mathsf{R}\, \psi_2$ if and only if, for all indexes $i \in [0, |\varpi|[$, it holds that $\varpi_{\geq i} \models \psi_2$ or there is an index $j \in [0, i[$ such that $\varpi_{\geq j} \models \psi_1$, and $|\varpi| = \omega$ or there is an index $j \in [0, |\varpi|[$ such that $\varpi_{\geq j} \models \psi_1$;
(8) $\varpi \models \tilde{\mathsf{X}}\, \psi$ if and only if $|\varpi| = 1$ or $\varpi_{\geq 1} \models \psi$;
(9) $\varpi \models \psi_1 \tilde{\mathsf{U}}\, \psi_2$ if and only if there is an index $i \in [0, |\varpi|[$ such that $\varpi_{\geq i} \models \psi_2$ and, for all indexes $j \in [0, i[$, it holds that $\varpi_{\geq j} \models \psi_1$, or $|\varpi| < \omega$ and, for all indexes $j \in [0, |\varpi|[$, it holds that $\varpi_{\geq j} \models \psi_1$;
(10) $\varpi \models \psi_1 \tilde{\mathsf{R}}\, \psi_2$ if and only if, for all indexes $i \in [0, |\varpi|[$, it holds that $\varpi_{\geq i} \models \psi_2$ or there is an index $j \in [0, i[$ such that $\varpi_{\geq j} \models \psi_1$.

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