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Abstract. In modal logics, *graded (world) modalities* have been deeply investigated as a useful framework for generalizing standard existential and universal modalities, in such a way they can express statements about a given number of immediately accessible worlds. These modalities have been recently investigated with respect to the μ-calculus, which have provided succinctness, without affecting the satisfiability of the extended logic, i.e., it remains solvable in EXPTIME. A natural question that arises is how logics that allow reasoning about paths could be effected by considering *graded path modalities*. In this paper, we investigate this question in the case of the branching-time temporal logic CTL and we call the extended logic GCTL, for short. We show several results for this logic. Among the others we prove that, although GCTL is more expressive than CTL, the satisfiability problem for GCTL remains solvable in EXPTIME. This result is obtained by exploiting an automata-theoretic approach. In particular, we introduce the class of *partitioning alternating Büchi tree automata* and show that the emptiness problems for them is EXPTIME-COMPLETE.

1 Introduction

Temporal logics are a special kind of *modal logics* that provide a formal framework for qualitatively describing and reasoning about how the truth values of assertions change over time. First pointed out by Pnueli in 1977 [Pnu77], these logics turn out to be particularly suitable for reasoning about correctness of concurrent programs [Pnu81].

Depending on the view of the underlying nature of time, two types of temporal logics are mainly considered [Lam80]. In *linear-time temporal logics*, such as LTL [Pnu77], time is treated as if each moment in time has a unique possible future. Conversely, in *branching-time temporal logics*, such as CTL [CE81] and CTL^* [EH86], each moment in time may split into various possible futures and *existential* and *universal quantifiers* are used to express properties along one or all the possible futures. In modal logics, such as $\mathcal{ALC}$ [SSS91] and μ-calculus [Koz83], these kinds of quantifiers have been generalized by means of *graded (worlds) modalities* [Fin72, Tob01], which allow to express properties such as "there exist at least n accessible worlds satisfying a certain formula" or "all but n accessible worlds satisfy a certain formula". For example, in a multitasking scheduling specification, we can express properties such as every time a computation is invoked, immediately next there are at least two spaces available for the allocation of two tasks that take care of the computation, without expressing exactly which spaces they are. This generalization has been proved to be very powerful as it allows to express system specifications in a very succinct way. In some cases, the extension makes the logic much more complex. An example is the guarded fragment of first order logic, which becomes undecidable when extended with a very weak form of counting quantifiers [Grä99]. In some other cases, one can extend a logic with very strong forms of "counting quantifiers" without increasing the computational complexity of the obtained logic. For example, this is the case for $\mu\mathcal{ALCQ}$ (see [BCM$^+$03] for a recent handbook) and graded μ-calculus [KSV02, BLMV06], for which the decidability problem is EXPTIME-COMPLETE.

Despite its high expressive power, the μ-calculus is considered in some sense a low-level logic, making it an "unfriendly" logic for users, whereas simpler logics, such as CTL, can naturally express complex properties of computation trees. Therefore, an interesting and natural question that arises is how the extension of CTL with graded modalities can affect its expressiveness and decidability. There is a technical challenge involved in such an extension, which makes this task non trivial, as instead one may think: in the μ-calculus, and other modal logics studied in the graded context so far, the existential and universal quantifiers range over the set of successors, thus it is easy to count the domain and its elements. In CTL, on the other hand, the underlying objects are both states and paths. Thus, the concept of graded must relapse on both of them. We solve this problem by introducing *graded path modalities* that extend to (*minimal* and *conservative*) paths the generalization induced to successor worlds by classical graded modalities, i.e., they allow to express properties such as "there are at least n minimal and conservative paths satisfying a formula", for a suitable and well-found concepts of minimality and conservativeness among paths. We call the logic CTL extended with graded path modalities GCTL, for short. The minimality property allows to decide GCTL formulas on a restricted but significant space domain, in a very natural way. In more details, it is enough to consider only the part of a system behavior that is effectively responsible of the satisfiability of a given formula, whenever each of its extensions satisfies the formula as well. So, we only take into account a set of non comparable paths satisfying the same property using in practice a particular equivalence relation on the set of all paths. Moreover, the minimality allows the graded path modalities to subsume the graded world modalities introduced for the μ-calculus. Indeed, if we drop the minimality, it makes no sense to discuss about the existence of a path in a structure, where the existence of a non minimal path satisfying a formula may induce also the existence of an infinite number of paths satisfying it.

With GCTL it is possible to express properties on a number of (not immediate) successor worlds, in a very succinct way, without explicitly stating properties on the intermediate worlds. As an example, consider the property "in a tree, there exists a path in which everytime p holds, n following grandchildren satisfy q". This property can be easily expressed in GCTL (linearly in n). Conversely, a graded μ-calculus formula would require to consider all possible children scenarios (i.e., all partitions of node successors) of p, and therefore it needs a length exponential in n. We also argue that this exponential blow-up is unavoidable. The ability of GCTL of reasoning about number of paths turns out to be suitable also to query XML documents. These documents, indeed, can be viewed as labeled unranked trees and GCTL allows to reasoning on a number of links among tags of descendant nodes, without naming any of the intermediate ones, in a very succinct way. As another and more practical example, consider again the above multitasking scheduling, where we may want to check that every time a non elementary (i.e., non one-step) computation is required, then there are at least n distinct (i.e., non completely equivalent) computational flows that can be executed. Again, this property can be easily expressed in GCTL. Furthermore, one may note that our framework of graded path quantifiers has some similarity with the concept of *cyclomatic complexity*, as it was defined by McCabe in a seminal work in software engineering [McC76]. McCabe studied a way to measure the complexity of a program, identifying it in the number of independent instruction flows. From an intuitive point of view, since graded path quantifiers allow to specify how many minimal computational paths satisfying a given property reside in a program, GCTL subsumes the cyclomatic complexity, where for independent we replace minimal.

The introduced framework of graded path modalities turns out to be very efficient in terms of expressiveness and complexity. Indeed, we prove that GCTL is more expressive than CTL, it retains the tree and the finite model properties, and its satisfiability problem is solvable in EXPTIME, therefore not harder than that for CTL [EH85]. The upper bound for the latter is obtained by exploiting an automata-theoretic approach [VW86, KVW00]. To develop a decision procedure for a logic with the tree model property, one first develops an appropriate notion of tree automata and studies their emptiness problem, then the satisfiability problem for the logic is reduced to the emptiness problem of the automata. To this aim, we introduce a new automata model: *partitioning alternating tree automata* (PATA). While a nondeterministic automaton on visiting a node of the input tree sends exactly one copy of itself to each successor of the node, an alternating automaton can send several copies of itself to the same successor. In particular, in *symmetric alternating automata* [JW95, Wil99] it is not necessary to specify the direction of the tree on which a copy is sent. In [KSV02], *graded alternating tree automata* (GATA) are introduced as a generalization of symmetric alternating tree automata, in such a way that the automaton can send copies of itself to a given number n of state successors, either in existential or universal way, without specifying which successors these exactly are. PATA further extend GATA in such a way that the automaton can send copies of itself to a given number n of paths. As we show later, for each GCTL formula φ, it is always possible to build in linear time a PATA along with a Büchi condition (PABT) $\mathcal{A}_\varphi$ accepting all the tree models of φ. The major difficulty here is that whenever φ contains graded modalities such as "there exist at least n minimal paths satisfying a path property ψ", $\mathcal{A}_\varphi$ must accept trees in which there are at least n distinct paths satisfying ψ, where some groups of those paths can arbitrarily share the same (proper) prefixes, and we ensure this, by constraining the transition relation of the automaton. We show an EXPTIME decision procedure for the emptiness of PABT through an exponential translation into non-deterministic Büchi tree automata (NBT). In more detail, we use a technical variation of the Miyano and Hayashi technique [MH84] for tree automata [Mos84], which has been deeply used in the literature for translating alternating Büchi automata (on both words and trees) to nondeterministic ones. Then, the result follows from the fact that the emptiness problem for NBT is solvable in polynomial time [VW86].

2 Preliminaries

Given a *set* X of *objects* (numbers, words, sets, etc.), we denote by $|\mathrm{X}|$ the number of its elements, called *size* of X, and by 2^{X} the *powerset* of X itself. In addition, by X^n we denote the set of all n-tuples of elements from X, by $\mathrm{X}^* = \bigcup_{n=0}^{\omega} \mathrm{X}^n$ the set of *finite words* on the *alphabet* X, and by X^+ the set $\mathrm{X}^* \setminus \{\varepsilon\}$, where, as usual, ω is the *numerable infinity* and ε is the *empty word*. By $|x|$ we denote the *length* of a word $x \in \mathrm{X}^*$ and by $\{x_i\}_i^n$ the *ordered sequence* $(x_1, \ldots, x_n) \in \mathrm{X}^+$ of objects varying on the index i. As special sets, we also consider $\mathbb{N}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ as, respectively, the sets of *natural numbers* and *positive natural numbers*. Furthermore, by $\mathbb{N}_{(n)}$ and $\mathbb{N}_{(n)+}$ we denote the subsets $\{k \in \mathbb{N} \mid k \leq n\}$ of $\mathbb{N}$ and $\{k \in \mathbb{N}_+ \mid k \leq n\}$ of $\mathbb{N}_+$, where $n \in \mathbb{N} \cup \{\omega\}$.

A *structure* $\mathcal{S}$ is an ordered tuple $\langle \mathrm{X}, \mathrm{R} \rangle$, where *(i)* $\mathrm{X} = \mathsf{dom}(\mathcal{S})$ is a non-empty set of objects, called *domain* of $\mathcal{S}$, and *(ii)* $\mathrm{R} \subseteq \mathrm{X} \times \mathrm{X}$ is a *binary relation* between objects. We denote the size $|\mathcal{S}|$ of $\mathcal{S}$ as the number $|\mathrm{X}|$ of objects of its domain. An infinite structure is a structure of infinite size. When the relation R is clear from the context, to refer to a structure we only use its domain. A *tree* is a structure $\langle \mathrm{X}, \mathrm{R} \rangle$ in which the domain X, in the following also referred as set of *nodes*, is a subset of $\mathbb{N}^*$ such that *(i)* if $x \cdot a \in \mathrm{X}$, with $x \in \mathbb{N}^*$ and $a \in \mathbb{N}$, then also $x \in \mathrm{X}$ and *(ii)* $(x, x') \in \mathrm{R}$ iff $x' = x \cdot a$, for some $a \in \mathbb{N}$. The empty word ε is the *root* of the tree. A tree is *full* iff $x \cdot a \in \mathrm{X}$, with $a \in \mathbb{N}$, implies $x \cdot b \in \mathrm{X}$, for all $b \in \mathbb{N}_{(a)}$. A *path* is a tree $\langle \mathrm{X}, \mathrm{R} \rangle$ in which for all nodes $x \in \mathrm{X}$ there is at most one $a \in \mathbb{N}$ such that $x \cdot a \in \mathrm{X}$, i.e., the transitive closure of the relation R is a linear (total) order on X. A Σ-*labeled* structure $\mathcal{S} = \langle \Sigma, \mathrm{X}, \mathrm{R}, \mathrm{L} \rangle$ is a tuple in which *(i)* Σ is a finite set of *labels*, *(ii)* $\langle \mathrm{X}, \mathrm{R} \rangle$ is a structure, and *(iii)* $\mathrm{L} : \mathrm{X} \mapsto \Sigma$ is a *labeling function* that colors each object with a label. When both Σ and R are clear from the context, we indicate a labeled structure $\langle \Sigma, \mathrm{X}, \mathrm{R}, \mathrm{L} \rangle$ with the shorter tuple $\langle \mathrm{X}, \mathrm{L} \rangle$.

Let $\mathcal{S} = \langle \mathrm{X}, \mathrm{R} \rangle$ and $\mathcal{S}' = \langle \mathrm{X}', \mathrm{R}' \rangle$ be two structures. We say that $\mathcal{S}'$ is a *substructure* of $\mathcal{S}$, in symbols $\mathcal{S}' \preccurlyeq \mathcal{S}$, iff *(i)* $\mathrm{X}' \subseteq \mathrm{X}$ and *(ii)* $\mathrm{R}' = \mathrm{R} \cap (\mathrm{X}' \times \mathrm{X}')$ hold. Moreover, we say that $\mathcal{S}$ and $\mathcal{S}'$ are *comparable* iff *(i)* $\mathcal{S} \preccurlyeq \mathcal{S}'$ or *(ii)* $\mathcal{S}' \preccurlyeq \mathcal{S}$ holds, otherwise they are *incomparable*. For a set of structures $\mathfrak{S}$, we define the set of *minimal substructures* $\mathsf{minstructs}(\mathfrak{S})$ as the set containing all and only the structures $\mathcal{S} \in \mathfrak{S}$ such that for all $\mathcal{S}' \in \mathfrak{S}$, it holds that *(i)* $\mathcal{S} \preccurlyeq \mathcal{S}'$, or *(ii)* $\mathcal{S}'$ is incomparable with $\mathcal{S}$. Note that all structures in $\mathsf{minstructs}(\mathfrak{S})$ are pairwise incomparable. A structure $\mathcal{S}$ is *minimal* w.r.t. a set $\mathfrak{S}$ (or simply minimal, when the context clarify the set $\mathfrak{S}$) iff $\mathcal{S} \in \mathsf{minstructs}(\mathfrak{S})$. A set of structures $\mathfrak{S}$ is minimal iff $\mathfrak{S} = \mathsf{minstructs}(\mathfrak{S})$.

A *Kripke structure* $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ is a 2^{AP}-labeled structure, where AP is a set of *atomic propositions*, $\mathrm{W} = \mathsf{dom}(\mathcal{K})$ is a set of *worlds* domain of the structure, R is a relation on W, and $\mathrm{L} : \mathrm{W} \mapsto 2^{\mathrm{AP}}$ is the labeling function that maps each world to a set of atomic propositions true in that world. Given a Kripke structure $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ and a world $w \in \mathrm{W}$, we define the *unwinding* of the structure $\mathcal{K}$ starting from w as the full and possibly infinite 2^{AP}-labeled (Kripke) tree $\mathcal{U}_w^{\mathcal{K}} = \langle \mathrm{AP}, \mathrm{W}', \mathrm{R}', \mathrm{L}' \rangle$ such that there is a function $\mathsf{uf} : \mathrm{W}' \mapsto \mathrm{W}$, called *unwinding function*, satisfying the following properties: *(i)* $\mathsf{uf}(\varepsilon) = w$ and, for all $w', v' \in \mathrm{W}'$ and $u \in \mathrm{W}$, it holds that *(ii)* $\mathrm{L}'(w') = \mathrm{L}(\mathsf{uf}(w'))$, *(iii)* if $(w', v') \in \mathrm{R}'$, then $(\mathsf{uf}(w'), \mathsf{uf}(v')) \in \mathrm{R}$, and, *(iv)* if $(\mathsf{uf}(v'), u) \in \mathrm{R}$, then there is one and only one $u' \in \mathrm{W}'$ such that $\mathsf{uf}(u') = u$ and $(v', u') \in \mathrm{R}'$. Note that the unwinding function, and so the unwinding structure, is unique up to *isomorphisms*. Given a Kripke structure $\mathcal{K}$ and a world $w \in \mathrm{W} = \mathsf{dom}(\mathcal{K})$, we define $\mathsf{paths}(\mathcal{K}, w)$ as the set of paths of

$\mathcal{K}$ starting from w. Formally, a path π is in $\mathsf{paths}(\mathcal{K}, w)$ iff $\pi \preccurlyeq \mathcal{U}_w^{\mathcal{K}}$. In addition, we set $\mathsf{paths}(\mathcal{K}) = \bigcup_{w \in \mathrm{W}} \mathsf{paths}(\mathcal{K}, w)$. With $\pi(\cdot)$ we denote the function $\pi : \mathbb{N}_{(|\pi|-1)} \mapsto \mathrm{W}$ that maps each number $k \in \mathbb{N}_{(|\pi|-1)}$ with the world $\pi(k) = \mathsf{uf}(w')$ of $\mathcal{K}$, which corresponds to the $(k+1)$-st position on the path π, where uf is the unwinding function relative to $\mathcal{U}_w^{\mathcal{K}}$, $w' \in \mathsf{dom}(\pi)$, and $|w'| = k$. Note that $\pi(0) = \mathsf{uf}(\varepsilon) = w$.

Finally, let $n \in \mathbb{N}_+$, we define the following two sets: $\mathsf{P}(n)$, as the set of all *solutions* $\{p_i\}_i^n$ to the *linear Diophantine equation* $1 * p_1 + 2 * p_2 + \ldots + n * p_n = n$ and $\mathsf{CP}(n)$ as the set of the *cumulative solutions* $\{cp_i\}_i^n$ obtained by summing increasing sets of elements from $\{p_i\}_i^n$. Formally, $\mathsf{P}(n) = \{\{p_i\}_i^n \in \mathbb{N}^n \mid \sum_{i=1}^n i * p_i = n\}$ and $\mathsf{CP}(n) = \{\{cp_i\}_i^n \in \mathbb{N}^n \mid \exists \{p_i\}_i^n \in \mathsf{P}(n)\, \forall i \in \mathbb{N}_{(n)+} : cp_i = \sum_{j=i}^n p_j\}$. Note that $|\mathsf{CP}(n)| = |\mathsf{P}(n)|$ and, since for each solution $\{p_i\}_i^n$ of the above Diophantine equation there is exactly one partition of n, it holds that $|\mathsf{CP}(n)| = p(n)$, where $p(n)$ is the number of partitions of n. Now, by a classical estimation of $p(n)$ due to Hardy and Ramanujan [Apo76], we know that, for a constant α, $p(n) = \Theta(\frac{1}{n} 2^{\alpha\sqrt{n}})$, so it follows that $|\mathsf{CP}(n)| = \Theta(\frac{1}{n} 2^{\alpha\sqrt{n}})$.

3 The Graded CTL temporal logic

In this section, we introduce an extension of the classical branching-time temporal logics CTL with graded path quantifiers. We show that this extension allows to gain expressiveness without paying any extra cost on deciding its satisfiability. For technical convenience, we introduce this logic through the state and path framework of CTL^*.

The *graded computation tree logic* (GCTL^*) extends CTL^* by using two special path quantifiers, the universal $\mathsf{A}^{<g}$ and the existential $\mathsf{E}^{\geq g}$, where g denotes the corresponding *degree*. As in CTL^*, these quantifiers can prefix a linear time formula composed by an arbitrary combination and nesting of the temporal operators X (*"effective next"*), $\tilde{\mathsf{X}}$ (*"hypothetical next"*), U (*"until"*), and R (*"release"*). The quantifiers $\mathsf{A}^{<g}$ and $\mathsf{E}^{\geq g}$ can be respectively read as *"all but less than g minimal paths"* and *"there exist at least g minimal paths"*. The formal syntax of GCTL^* follows.

Definition 1. (Syntax) *$GCTL^*$* state *(φ) and* path *(ψ)* formulas *are built inductively from* AP *using the following context-free grammar, where $p \in \mathrm{AP}$ and $g \in \mathbb{N}$:*

1. $\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{A}^{<g}\psi \mid \mathsf{E}^{\geq g}\psi$,
2. $\psi ::= \varphi \mid \neg\psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \mathsf{X}\,\psi \mid \tilde{\mathsf{X}}\,\psi \mid \psi\,\mathsf{U}\,\psi \mid \psi\,\mathsf{R}\,\psi$.

The class of $GCTL^*$ formulas is the set of state formulas generated by the above grammar. In addition, the simpler class of GCTL formulas is obtained by forcing each temporal operator, occurring into a formula, to be coupled with a path quantifier as in the classical definition of CTL.

For a state formula φ, we define the *degree* $\mathsf{deg}(\varphi)$ of φ as the maximum natural number g occurring among the degrees of all its path quantifiers. We assume that all such degrees are coded in unary. Accordingly, the *length* of a formula φ, denoted by $|\varphi|$, is defined inductively on the structure of φ itself in a classical way, and by also considering $|\mathsf{A}^{<g}\psi|$ and $|\mathsf{E}^{\geq g}\psi|$ to be equal to $g+1+|\psi|$. It is obvious that $\mathsf{deg}(\varphi) = O(|\varphi|)$.

We now define the semantics of GCTL^* w.r.t. a Kripke structure $\mathcal{K}$. For a world $w \in \mathsf{dom}(\mathcal{K})$, we write $\mathcal{K}, w \models \varphi$ to indicate that a state formula φ holds at w, and, for a path $\pi \in \mathsf{paths}(\mathcal{K})$, we write $\mathcal{K}, \pi, k \models \psi$ to indicate that a path formula ψ holds on π at position $k \in \mathbb{N}_{(|\pi|-1)}$. Note that, the relation $\mathcal{K}, \pi, k \models \psi$ does not hold for any point $k \in \mathbb{N}$, with $k \geq |\pi|$. For a better readability, in the semantics definition of GCTL^* we use the special set $\mathfrak{P}_A(\mathcal{K}, w, \psi)$ and its dual $\mathfrak{P}_E(\mathcal{K}, w, \psi)$, with the following meaning: $\mathfrak{P}_A(\mathcal{K}, w, \psi)$ contains every path π starting in w such that all its extensions π' (including π) satisfy the path formula ψ. The semantics of GCTL^* is formally defined as follows.

Definition 2. (Semantics) *Given a Kripke structure $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ and $w \in \mathrm{W}$, for all GCTL^* state formulas φ, the relation $\mathcal{K}, w \models \varphi$, is inductively defined as follows.*

1. $\mathcal{K}, w \models p$, *with* $p \in \mathrm{AP}$, *iff* $p \in \mathrm{L}(w)$.
2. *For all state formulas φ, φ_1, and φ_2, it holds:*
   (a) $\mathcal{K}, w \models \neg\varphi$ *iff not* $\mathcal{K}, w \models \varphi$, *that is* $\mathcal{K}, w \not\models \varphi$;
   (b) $\mathcal{K}, w \models \varphi_1 \wedge \varphi_2$ *iff* $\mathcal{K}, w \models \varphi_1$ *and* $\mathcal{K}, w \models \varphi_2$;
   (c) $\mathcal{K}, w \models \varphi_1 \vee \varphi_2$ *iff* $\mathcal{K}, w \models \varphi_1$ *or* $\mathcal{K}, w \models \varphi_2$.
3. *For a path formula ψ and a natural number g, it holds:*
   (a) $\mathcal{K}, w \models \mathsf{A}^{<g}\psi$ *iff* $|\mathsf{minstructs}(\mathsf{paths}(\mathcal{K}, w) \setminus \mathfrak{P}_E(\mathcal{K}, w, \psi))| < g$;
   (b) $\mathcal{K}, w \models \mathsf{E}^{\geq g}\psi$ *iff* $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \psi))| \geq g$;
   where $\mathfrak{P}_A(\mathcal{K}, w, \psi) = \{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi'$ *implies* $\mathcal{K}, \pi', 0 \models \psi\}$ *and* $\mathfrak{P}_E(\mathcal{K}, w, \psi) = \{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \exists \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi'$ *and* $\mathcal{K}, \pi', 0 \models \psi\}$.

For all GCTL^* path formulas ψ, paths $\pi \in \mathsf{paths}(\mathcal{K})$, and natural numbers $k < |\pi|$, the relation $\mathcal{K}, \pi, k \models \psi$ is inductively defined as follows.

4. $\mathcal{K}, \pi, k \models \varphi$, *with φ state formula, iff* $\mathcal{K}, \pi(k) \models \varphi$.
5. *Where ψ, ψ_1, and ψ_2 are path formulas, we have:*
   (a) $\mathcal{K}, \pi, k \models \neg\psi$ *iff not* $\mathcal{K}, \pi, k \models \psi$, *that is* $\mathcal{K}, \pi, k \not\models \psi$;
   (b) $\mathcal{K}, \pi, k \models \psi_1 \wedge \psi_2$ *iff* $\mathcal{K}, \pi, k \models \psi_1$ *and* $\mathcal{K}, \pi, k \models \psi_2$;
   (c) $\mathcal{K}, \pi, k \models \psi_1 \vee \psi_2$ *iff* $\mathcal{K}, \pi, k \models \psi_1$ *or* $\mathcal{K}, \pi, k \models \psi_2$.
6. *Where ψ, ψ_1, and ψ_2 path formulas, we have:*
   (a) $\mathcal{K}, \pi, k \models \mathsf{X}\,\psi$ *iff* $k < |\pi| - 1$ *and* $\mathcal{K}, \pi, (k+1) \models \psi$;
   (b) $\mathcal{K}, \pi, k \models \tilde{\mathsf{X}}\,\psi$ *iff* $k = |\pi| - 1$ *or* $\mathcal{K}, \pi, (k+1) \models \psi$;
   (c) $\mathcal{K}, \pi, k \models \psi_1 \mathsf{U}\, \psi_2$ *iff there is an index i, with $k \leq i < |\pi|$, such that* $\mathcal{K}, \pi, i \models \psi_2$ *and, for all indexes j with $k \leq j < i$, it holds* $\mathcal{K}, \pi, j \models \psi_1$;
   (d) $\mathcal{K}, \pi, k \models \psi_1 \mathsf{R}\, \psi_2$ *iff for all indexes i, with $k \leq i < |\pi|$, it holds* $\mathcal{K}, \pi, i \models \psi_2$ *or there is an index j with $k \leq j < i$, such that* $\mathcal{K}, \pi, j \models \psi_1$.

Note that GCTL^* (resp., GCTL) formulas with degrees 1 are CTL^* (resp., CTL) formulas. Moreover, the above definition of $\mathfrak{P}_A(\mathcal{K}, w, \psi)$ and $\mathfrak{P}_E(\mathcal{K}, w, \psi)$, formally states that they are dual of each other, i.e., $\mathfrak{P}_A(\mathcal{K}, w, \psi) = \mathsf{paths}(\mathcal{K}, w) \setminus \mathfrak{P}_E(\mathcal{K}, w, \neg\psi)$.

For all state formulas φ_1 and φ_2 (resp., path formulas ψ_1 and ψ_2), we say that φ_1 is *equivalent* to φ_2, formally $\varphi_1 \equiv \varphi_2$, (resp., ψ_1 is *equivalent* to ψ_2, formally $\psi_1 \equiv \psi_2$) iff for all Kripke structures $\mathcal{K}$ and worlds $w \in \mathsf{dom}(\mathcal{K})$ it holds that $\mathcal{K}, w \models \varphi_1$ iff $\mathcal{K}, w \models \varphi_2$ (resp., $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \psi_1)) = \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \psi_2))$).

In the rest of the paper, we only consider formulas in *existential normal form* or in *positive normal form*, i.e., formulas in which only existential quantifiers occur or negation is applied only to atomic propositions, respectively. In fact, it is to this aim that we have considered in the syntax of GCTL^* both the connectives $\wedge$ and $\vee$, the quantifiers $\mathsf{A}^{<g}$ and $\mathsf{E}^{\geq g}$, and the dual operators $\tilde{\mathsf{X}}$ and R. Indeed, all formulas can be converted in existential or positive normal form by using De Morgan's laws and the following equivalences, which directly follow from the semantics of the logic. Let ψ, ψ_1, and ψ_2 be path formulas and $g \in \mathbb{N}$, it holds that $\neg \mathsf{A}^{<g}\psi \equiv \mathsf{E}^{\geq g}\neg\psi$, $\neg \mathsf{X}\,\psi \equiv \tilde{\mathsf{X}}\,\neg\psi$, and $\neg(\psi_1 \mathsf{U}\, \psi_2) \equiv \neg\psi_1 \mathsf{R}\, \neg\psi_2$. In order to abbreviate writing formulas we also use the boolean values $\mathfrak{t}$ (*"true"*) and $\mathfrak{f}$ (*"false"*) and the path quantifiers $\mathsf{E}\psi \equiv \mathsf{E}^{\geq 1}\psi$ (*"there is a minimal path"*) and $\mathsf{E}^{>g}\psi \equiv \mathsf{E}^{\geq g+1}\psi$ (*"there are more than one minimal path"*).

The following lemma shows interesting equivalences among GCTL formulas that will be useful to show important properties of the introduced logic. In particular, we show fixed point equivalences that extend to "graded" formulas the well known analogous ones for "ungraded" formulas.

Lemma 1. *For all state formulas* φ_1 *and* φ_2 *and degrees* $g > 1$*, it holds that:*

$$i \begin{cases} \mathsf{E}(\varphi_1 \mathsf{U}\, \varphi_2) \equiv \varphi_2 \vee \varphi_1 \wedge \mathsf{ex}(\varphi_1 \mathsf{U}\, \varphi_2, 1) \\ \mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2) \equiv \neg\varphi_2 \wedge \varphi_1 \wedge \mathsf{ex}(\varphi_1 \mathsf{U}\, \varphi_2, g) \end{cases}$$

$$ii \begin{cases} \mathsf{E}(\varphi_1 \mathsf{R}\, \varphi_2) \equiv \varphi_2 \wedge (\varphi_1 \vee \mathsf{E}\tilde{\mathsf{X}}\, \mathfrak{f} \vee \mathsf{ex}(\varphi_1 \mathsf{R}\, \varphi_2, 1)) \\ \mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2) \equiv \varphi_2 \wedge \neg\varphi_1 \wedge \mathsf{EX}\, \mathsf{E}\neg(\varphi_1 \mathsf{R}\, \varphi_2) \wedge \mathsf{ex}(\varphi_1 \mathsf{R}\, \varphi_2, g) \end{cases}$$

where $\mathsf{ex}(\psi, g) = \bigvee_{\{h_i\}_i^g \in \mathsf{CP}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\, \mathsf{E}^{\geq i}\psi$.

The function $\mathsf{ex}(\psi, g)$ used in the above lemma allows to partition g paths through h_1 successor worlds, for a given sequence $\{h_i\}_i^g \in \mathsf{CP}(g)$. Indeed, h_i is the number of successor worlds from which at least i paths satisfying ψ start. Therefore, h_1 is a right bound on the number of successor worlds we have to consider to ensure the satisfiability of the formula. By a simple calculation, it also follows that $|\mathsf{ex}(\psi, g)| = g * (|\psi| + \frac{g+11}{2}) * |\mathsf{CP}(g)| - 1 = \Theta((|\psi| + \frac{g}{2}) * 2^{\alpha\sqrt{g}})$, for a constant α. Note that, for $g = 1$, Lemma 1 gives the two classical fixed point expansions for CTL: $\mathsf{E}(\varphi_1 \mathsf{U}\, \varphi_2) \equiv \varphi_2 \vee \varphi_1 \wedge \mathsf{EX}\, \mathsf{E}(\varphi_1 \mathsf{U}\, \varphi_2)$ and $\mathsf{E}(\varphi_1 \mathsf{R}\, \varphi_2) \equiv \varphi_2 \wedge (\varphi_1 \vee \mathsf{E}\tilde{\mathsf{X}}\, \mathfrak{f} \vee \mathsf{EX}\, \mathsf{E}(\varphi_1 \mathsf{R}\, \varphi_2))$.

Let $\mathcal{K}$ be a Kripke structure and φ be a GCTL^* formula. Then, $\mathcal{K}$ is a *model* for φ, denoted by $\mathcal{K} \models \varphi$, iff there is $w \in \mathsf{dom}(\mathcal{K})$ such that $\mathcal{K}, w \models \varphi$. In this case, we also say that $\mathcal{K}$ is a model for φ on w. A GCTL^* formula φ is said *satisfiable* iff there exists a model for it, moreover it is *invariant* on the two Kripke structures $\mathcal{K}$ and $\mathcal{K}'$ iff either $\mathcal{K} \models \varphi$ and $\mathcal{K}' \models \varphi$ or $\mathcal{K} \not\models \varphi$ and $\mathcal{K}' \not\models \varphi$.

In the next theorem, we show an exponential reduction of GCTL to the graded μ-calculus[1].

Theorem 1. *Given a GCTL formula* φ *there exists an equivalent graded* μ*-calculus formula* φ' *whose size is at most exponential in the size of* φ.

By using the fact that for graded μ-calculus the satisfiability problem is solvable in EXPTIME [KSV02], we immediately get that the satisfiability problem for GCTL is decidable and solvable in 2EXPTIME, as reported in the following corollary. However,

[1] The μ-calculus is a well-known modal logic augmented with fixed point operators [Koz83]. The graded μ-calculus extends the μ-calculus with graded state quantifiers [KSV02, BLMV06].

in the next section we improve this result by showing that the satisfiability problem for GCTL is solvable in EXPTIME, by exploiting an automata-theoretic approach that deeply makes use of the idea behind the function $\mathsf{ex}(\psi, g)$ introduced in Lemma 1.

Corollary 1. *The satisfiability problem for GCTL is decidable in 2*EXPTIME*.*

We conclude this section by showing some interesting properties about GCTL. First of all, by using a proof by induction we show that this logic is invariant under the unwinding of a model. Directly from this, we get that GCTL also enjoys the tree model property. Moreover, by extending a technique introduced in [EH85] along with Lemma 1, for each GCTL formula φ it is possible to build an Hintikka structure from which we can get a finite model[2] for φ. By means of a counterexample, we can also show that GCTL is not invariant under bisimulation among models. Directly from this, we obtain that GCTL is more expressive than CTL, since the latter is invariant under bisimulation. All these properties are reported in the following theorem.

Theorem 2. *For GCTL it holds that* (i) *it is invariant under unwinding;* (ii) *it has the tree model property;* (iii) *it has the finite model property;* (iv) *is not invariant under bisimulation; and* (v) *it is more expressive than CTL.*

4 Partitioning Büchi Tree Automata

Nondeterministic automata on infinite trees are an extension of nondeterministic automata on infinite words and finite trees (see [Tho90] for an introduction). *Alternating automata* [MS87] are a generalization of nondeterministic automata that embody the same concept of alternation as Turing machines [CKS81]. Intuitively, while a nondeterministic automaton that visits a node of the input tree sends exactly one copy of itself to each of the successors of the node, an alternating automaton can send several copies of itself to the same successor. *Symmetric automata* [JW95, Wil99] are a variation of classical (asymmetric) alternating automata in which it is not necessary to specify the direction of the tree on which a copy is sent. In fact, through three generalized directions (ε-moves, existential moves, and universal moves), it is possible to send a copy of the automaton, starting from a node of the input tree, to the same node, to some of its successors, or to all its successors. Hence, the automaton does not distinguish between directions. As a generalization of symmetric automata, *graded alternating tree automata* (*GATA*, for short) have also been introduced [KSV02]. In this framework, the automaton can send copies of itself to a given number n of successors, either in existential or universal way, without specifying which successors these exactly are. Moreover, a GATA can also send a copy of itself to the reading node by pursuing an ε-move.

Here, we consider *partitioning alternating tree automata* (*PATA*, for short) as a generalization of GATA in such a way that the automaton can send copies of itself to a given number n of paths, starting from the current node. As we show later, for each GCTL formula φ, it is possible to built a PATA that accepts all and only the tree models of φ. The key idea is to extend GATA's runs by also labeling their nodes with a natural

[2] To the best of our knowledge, we do not know whether also the graded μ-calculus enjoys the finite model property.

number, with the aim of collecting “graded path information”. We give an idea on how a PATA $\mathcal{A}$ works w.r.t. the logic GCTL through an example.

First, note that $\mathcal{A}$ uses as states all possible subformulas of the considered formula[3]. Now, suppose that the automaton is in the node x of an input tree T and in state $\mathsf{E}^{\geq g}\psi$, where ψ is also a GCTL path formula. Then, in a state corresponding to ψ, the automaton sends $n \leq g$ copies of itself to n successors of x with degrees $g_1,\ldots,g_n$ that sum to g. One can note that this sequence of n degrees is a partition of the number g. The degree g_i associated to a successor x_i of x denotes that at least g_i paths starting from x_i have to satisfy ψ and the automaton takes care of it through the transition function. In more details, we individuate the set of n directions relative to successors of x w.r.t. the degrees $\{g_1,\ldots,g_n\}$ by means of a decreasing chain $\{M_1,\ldots,M_{n+1}\}$, such that for each i, it holds that $M_i \setminus M_{i+1}$ contains all directions of x that are associated with a degree i. Clearly, there could be different possible chains satisfying such a property and each one induces a different run of $\mathcal{A}$ on T. As a particular case, $\mathcal{A}$ sends g copies of itself to g distinct successors of x on choosing $|M_1| = g$ and, for each $i > 1$, $M_i = \emptyset$.

The formal definition of a PATA along with the Büchi acceptance condition follows. In particular, we give a definition without any constraint on the use of its labeling degrees, which allows to introduce a more general class of automata, independently from the logic we consider here. Note that by the definition we give, the automaton at its own cannot enforce that multiple successors in which it is sent are all distinct. However, we can force this by means of the transition function. First, we introduce some extra notation. With $\mathsf{B}^+(\mathrm{X})$ we denote the sets of *positive Boolean formulas* over X (i.e., Boolean formulas built from elements in X using $\wedge$ and $\vee$) where we also allow the formulas $\mathfrak{t}$ (true) and $\mathfrak{f}$ (false). For a set $\mathrm{X}' \subseteq \mathrm{X}$ and a formula $\phi \in \mathsf{B}^+(\mathrm{X})$, we say that X' satisfies ϕ, $\mathrm{X}' \models \phi$, iff the assigning of true to elements in X' and false to elements in $\mathrm{X} \setminus \mathrm{X}'$ makes ϕ true. With D_b and D_b^{ε} we denote the sets $\{\Diamond,\Box\} \times \mathbb{N}_{(b)+}$ and $D_b \cup \{\varepsilon\}$, respectively. Intuitively, these two sets represent the generalized directions that one can use, through the transition function, to describe the behavior of the automaton. For brevity, we often write $\langle g\rangle$ and $[g]$ instead of $(\Diamond, g)$ and $(\Box, g)$, respectively.

Definition 3. (PABT) *A* partitioning alternating Büchi tree automaton *is a tuple $\mathcal{A} = \langle Q,\Sigma,b,\delta,q_0,g_0,F\rangle$, where Q is a finite set of states, Σ is a finite input alphabet, $b \in \mathbb{N}$ is a counting branching bound, $\delta : Q \times \mathbb{N}_{(b)} \times \Sigma \mapsto \mathsf{B}^+(D_b^{\varepsilon} \times Q)$ is a transition function, $q_0 \in Q$ is an initial state, $g_0 \in \mathbb{N}$ is an initial branching degree, and $F \subseteq Q \times \mathbb{N}_{(b)}$ is a Büchi acceptance condition, which is defined later.*

The behavior of a PABT is described by means of a run. As for classical alternating automata, given a PABT $\mathcal{A} = \langle Q,\Sigma,b,\delta,q_0,g_0,F\rangle$ and a Σ-labeled tree $\langle \mathsf{T},\mathsf{inp}\rangle$ in input, a run $\langle \mathsf{T_r},\mathsf{run}\rangle$ of $\mathcal{A}$ on $\langle \mathsf{T},\mathsf{inp}\rangle$ is induced by the sets of pairs $S \subseteq D_b^{\varepsilon} \times Q$ satisfying its transition function δ. Here, we first give an intuition of such a run through an example. Suppose that $\mathcal{A}$, while reading a node x of T labeled with σ, is in a state q with degree g at the node y of the run, and $\delta(q,g,\sigma) = (\varepsilon,q_1) \wedge ((\langle 3\rangle,q_2) \vee ([2],q_3)$. Also, suppose that x has three successors $\{x\cdot 0, x\cdot 1, x\cdot 2\}$. Consider now $S = \{(\varepsilon,q_1),(\langle 3\rangle,q_2)\}$ satisfying $\delta(q,g,\sigma)$. Accordingly, $\mathcal{A}$ can send a copy of itself to node x in the state q_1 (by

[3] More precisely, the automaton uses as states an extended definition of the Fischer-Ladner. See proof of Theorem 3 for a formal definition.

performing an ε-move) and three copies of itself in the state q_2 to three paths through either one, two, or all successors of x. Now, suppose that we want to send two copies of $\mathcal{A}$ through one successor and one through another. This can be characterized by taking $M_1 = \{0,1\}$, $M_2 = \{1\}$, and $M_3 = M_4 = \emptyset$. Consequently, the run has three successors $\{y\cdot 0, y\cdot 1, y\cdot 2\}$, one labeled with $(x, q_1, 0)$ (for the ε-move), another labeled with $(x\cdot 0, q_2, 1)$, and the last one labeled with $(x\cdot 1, q_2, 2)$.

We now give the formal definition of a run. To this aim, we first formally define the sets $\{M_i\}_i^{g+1}$ introduced above, through a function spart, useful to define the required splitting among paths. Then, we introduce a function exec that allows to construct all possible execution steps.

Definition 4. (Splitting partition function) *A* splitting partition function $\mathsf{spart} : (D, d) \in 2^{\mathbb{N}} \times D_b \mapsto \mathsf{spart}(D,d) \in 2^{(2^{\mathbb{N}})^+}$ *maps a set D and a direction d into a set of decreasing chains* $\{M_i\}_i$ *of subset of D* ($M_i \subseteq D$ *and* $M_i \supseteq M_{i+1}$) *such that:*

1. *if* $d = \langle g \rangle$, *then for all* $\{M_i\}_i^{g+1} \in \mathsf{spart}(D,d) \subseteq (2^D)^{g+1}$, *it holds that* $M_{g+1} = \emptyset$ *and there is a sequence* $\{h_i\}_i^g \in \mathsf{CP}(g)$ *such that* $|M_j| = h_j$, *for all* $j \in \mathbb{N}_{(g)+}$;
2. *if* $d = [g]$, *then for all* $\{M_i\}_i^{g+1} \in \mathsf{spart}(D,d) \subseteq (2^D)^{g+1}$, *it holds that* $M_1 = D$ *and for all sequences* $\{h_i\}_i^g \in \mathsf{CP}(g)$ *there is* $j \in \mathbb{N}_{(g)+}$ *such that* $|M_{j+1}| < h_j$.

Differently form GATA, one can see that in general the functions $\mathsf{spart}(D, \langle g \rangle)$ and $\mathsf{spart}(D, [g])$ are not the dual of each other. This is due to the fact that in PATA, for a considered node x, we may want to check properties along paths starting in x, instead of just looking at the successors of x, as it is done in GATA. This induces, in the $d = \langle g \rangle$ case, to take care of just g paths (on which we check that a certain property holds), while in the $d = [g]$ case we have to take care of all paths (i.e., that in less than g paths the property may or may not hold, while in all the remaining ones it must hold).

We now give the formal definition of the function exec. Let $\mathbb{N}_\varepsilon$ denote the set $\mathbb{N} \cup \{\varepsilon\}$.

Definition 5. (Execution function) *An* execution function $\mathsf{exec} : (S, D) \in 2^{D_b^\varepsilon \times Q} \times 2^{\mathbb{N}_\varepsilon} \mapsto \mathsf{exec}(S,D) \in 2^{2^{\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}}}$ *maps the two sets S and D into the set of all possible subsets of* $\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}$, *called* configurations *of the execution, such that, for all sets* $E \in 2^{\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}}$ *we have* $E \in \mathsf{exec}(S,D)$ *iff for all pairs* $(d,q) \in S$ *it holds that:*

1. *if* $d = \varepsilon$ *then* $(\varepsilon, q, 0) \in E$;
2. *if either* $d = \langle g \rangle$ *or* $d = [g]$ *then there exists a sequence* $\{M_i\}_i^{g+1} \in \mathsf{spart}(D,d)$ *such that for all indexes* $i \in \mathbb{N}_{(g)+}$ *and direction* $x \in M_i \setminus M_{i+1}$, *it holds that* $(x,q,i) \in E$.

The above function exec allows us to give the following definition of PABT's run in a very concise and elegant way. First, we introduce the following extra notation. Let $\mathrm{X}' \subseteq \mathrm{X}^*$ be a set of words on X and $x \in \mathrm{X}^*$. Then, we denote by $\mathsf{succ}_{\mathrm{X}'}(x)$ the set of *successor words* of x in X', i.e., $\mathsf{succ}_{\mathrm{X}'}(x) = \{x \cdot a \in \mathrm{X}' \mid a \in \mathbb{N}\}$ and by $\mathsf{dir}_{\mathrm{X}'}(x)$ the set of *direction* of x in X', i.e., $\mathsf{dir}_{\mathrm{X}'}(x) = \{a \in \mathbb{N} \mid x \cdot a \in \mathrm{X}'\}$. Now, let $f : \mathrm{X}' \mapsto \mathrm{X}''$. We use $\mathsf{inf}(f)$ to refer to the set $\{x' \in \mathrm{X}' \mid |f^{-1}(x')| = \omega\}$, i.e., the set of elements of X' that f uses infinitely often as labels for elements in X, and $f_{|\mathrm{X}'''}$ to indicate the restriction of f to X''', i.e., $f_{|\mathrm{X}'''} : \mathrm{X}''' \mapsto \mathrm{X}''$, where $\mathrm{X}''' \subseteq \mathrm{X}'$. In the following we also write $S \models \delta(q,g,\sigma)$ to denote that S is a set of tuples $(d,q) \in D_b^\varepsilon \times Q$ that satisfies $\delta(q,g,\sigma)$.

Definition 6. (Run of a PABT) *A* run *of a PABT $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F\rangle$ on a Σ-labeled tree $\langle \mathsf{T}, \mathsf{inp}\rangle$ is a $(\mathsf{T} \times Q \times \mathbb{N}_{(b)})$-labeled full tree $\langle \mathsf{T_r}, \mathsf{run}\rangle$ such that:*

1. $\mathsf{run}(\varepsilon) = (\varepsilon, q_0, g_0)$;
2. *for all $y \in \mathsf{T_r}$ with $\mathsf{run}(y) = (x, q, g)$, there exist a set $S \subseteq D^{\varepsilon}_b \times Q$, where $S \models \delta(q, g, \mathsf{inp}(x))$, and a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathsf{T}}(x))$ such that for all configurations $(d, q', g') \in E$ there is a node $y' \in \mathsf{succ}_{\mathsf{T_r}}(y)$ such that $\mathsf{run}(y') = (x \cdot d, q', g')$.*

The run $\langle \mathsf{T_r}, \mathsf{run}\rangle$ is accepting *iff all its infinite paths satisfy the acceptance condition, i.e., for all paths $\pi \preccurlyeq \mathsf{T_r}$, with $|\pi| = \omega$, it holds that $\mathsf{inf}(\mathsf{run}_{|\pi}) \cap \mathsf{T} \times F \neq \emptyset$. A tree $\langle \mathsf{T}, \mathsf{inp}\rangle$ is accepted by $\mathcal{A}$ iff there is an accepting run of $\mathcal{A}$ on it. By $\mathcal{L}(\mathcal{A})$ we denote the language accepted by the automaton $\mathcal{A}$, i.e., the set of all input trees that $\mathcal{A}$ accepts. $\mathcal{A}$ is said to be empty if $\mathcal{L}(\mathcal{A}) = \emptyset$. The emptiness problem for $\mathcal{A}$ is to decide whether $\mathcal{L}(\mathcal{A}) = \emptyset$.*

By extending a construction given in [KVW00], we obtain the following result.

Theorem 3. *Given a GCTL formulas φ with degree b, we can construct in time $O(|\varphi|)$ a PABT $\mathcal{A}_\varphi$, with $O(|\varphi|)$ states and counting branching bound b, such that $\mathcal{L}(\mathcal{A}_\varphi)$ is exactly the set of all tree models of φ.*

Proof. (Sketch.) The automaton $\mathcal{A}_\varphi$ is defined as the tuple $\langle \mathsf{ecl}(\varphi), 2^{\mathrm{AP}}, \mathsf{deg}(\varphi), \delta, \varphi, 0, F\rangle$, where $\mathsf{ecl}(\varphi)$ is the *Fisher-Ladner* closure extended in order to deal with graded path modalities. The acceptance condition F is the set of all pairs $(\langle \varphi_1 \mathsf{R}\, \varphi_2\rangle, 1)$ and $([\varphi_1 \mathsf{R}\, \varphi_2], 1)$ of $\mathsf{ecl}(\varphi) \times \mathbb{N}_{(b)}$. The transition function extends that defined in [KVW00] for CTL, along with the extra graded path modalities. A formal proof of the correctness of the whole construction follows quite naturally, by extending that (by induction on the structure of the formula) used for CTL. Its formal definition and a detailed proof of its correctnessis are reported in Appendix D. Here, we only give an intuition of it through a couple of examples.

First, recall that δ is a function from $\mathsf{ecl}(\varphi) \times \mathbb{N}_{(b)} \times 2^{\mathrm{AP}}$ into $\mathsf{B}^+(D^{\varepsilon}_b \times \mathsf{ecl}(\varphi))$. Consider the state formula $\varphi = \mathsf{E}^{\geq g} \mathsf{X}\, \varphi'$. This formula is true on a tree model rooted at a node x having at least g distinct successors of x satisfying φ'. This is ensured through the δ in two successive steps. First, starting from the state $\mathsf{E}^{\geq g} \mathsf{X}\, \varphi'$, the δ gives the formula $(\langle g\rangle, \langle \varphi'\rangle)$, which intends to send to g successors (not necessarily distinct) the check of the satisfiability of φ'. Then, from state $\langle \varphi'\rangle$ we have to ensure that each of such successor nodes, say it y, contributes to the satisfiability of exactly one φ' (intuitively one degree of φ). Therefore, on reading y, if the degree associated with the state $\langle \varphi'\rangle$ is greater then 1, the δ returns false, otherwise, with an ε-move, we move to state φ'. Accordingly, in the δ we use as counting branching positive numbers to indicate formulas' degrees which have to be accomplished along paths and use as a convention 0 if we have none to accomplish. In particular, ε-moves always give 0 as counting branching.

As another example, consider the state formula $\varphi = \mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2)$. This formula is true on a tree model rooted at a node x having at least g distinct minimal paths satisfying $\varphi_1 \mathsf{U}\, \varphi_2$. As in CTL, the path formula $\varphi_1 \mathsf{U}\, \varphi_2$ is true on a path if φ_2 is immediately true, or φ_1 is immediately true and then the formula $\varphi_1 \mathsf{U}\, \varphi_2$ is satisfied on the successor node. Moreover, the quantifiers $\mathsf{E}^{\geq g}$ requires that there are at least g of such (minimal) paths. Therefore, if $g = 1$ the δ proceeds as in CTL. Conversely, if $g > 1$ we have to force φ_2 do not be immediately true (otherwise, we have less than g paths satisfying the formula). Thus, we use the δ to ensure that φ_1 is immediately true and that g successive paths (but not necessarily all distinct) satisfy $\varphi_1 \mathsf{U} \varphi_2$. Iteratively, the δ keeps using the above idea up to all states corresponding to the formula $\varphi_1 \mathsf{U} \varphi_2$ are sent to next nodes with counting branching 1. This ensures that the considered tree model has at least g minimal paths satisfying the formula $\varphi_1 \mathsf{U} \varphi_2$. Note that if less than g of such paths exist in the tree model, then the automaton keeps regenerating infinitely often the state corresponding to the until formula. Such a tree is then not accepting as this state is not in F. It is worth noting that, the above iteration upon the until states inherits the fixed point idea of the function $\mathsf{ex}(\psi, g)$ introduced in Lemma 1. In particular, we formally embed it into the δ through the formula $(\langle 1 \rangle, \langle \varphi_1 \mathsf{U} \varphi_2 \rangle)$ (see the formal definition of δ in appendix for details). This is a key step in our construction, since it allows to treat the exponential blow-up induced by the mentioned function by only using a constant rule into the δ.

In the remaining part of this section, we illustrate how the emptiness problem for PABT can be solved in ExpTime. To gain this result, we use a technical extension of the Miyano and Hayashi technique [MH84] for tree automata [Mos84], which has been deeply used in the literature for translating asymmetric alternating Büchi automata to nondeterministic ones in exponential time. Here, we use this technique to translate with the same blow-up a PABT into nondeterministic Büchi tree automata (NBT, for short). Roughly speaking, this means that we manage to combine the exponential blow-up induced by the alternation and that induced by the permutations of all possible splitting degrees in only one exponential blow-up. This technique is illustrated in the next theorem.

Theorem 4. *Let $\mathcal{A}$ be a PABT with n states and counting branching bound b. Then, there exists a NBT $\mathcal{A}'$ with $2^{2n*(b+1)}$ states and $n*b(b+1)/2$ directions such that $\mathcal{A}$ is not empty iff $\mathcal{A}'$ is not.*

Proof. (Sketch.) The NBT $\mathcal{A}'$ guesses a subset construction applied to a run of the PABT. At a given node x of a run of $\mathcal{A}'$, it keeps in its memory the set of states in which the various copies of $\mathcal{A}$ visit the node x in the guessed run. In order to make sure that every infinite path visits accepting states in F infinitely often, $\mathcal{A}'$ keeps track of states that “owe” a visit to F. The fact that PABT are symmetric, however, requires further non-trivial work, indeed, differently from the classical approach, we have to convert the symmetric automaton $\mathcal{A}$ into a nondeterministic one. This is because, while for symmetric automata there is bijective correspondence between direction of both the input and output automaton, in our case we have to build this correspondence by looking at the δ of the input automaton. The extra problems are: *(i)* $\mathcal{A}$ can perform ε-moves and *(ii)* $\mathcal{A}$ does not have an upper bound on the number of directions it uses. The first problem is solved by using in $\mathcal{A}'$ an apposite direction that collects all states of $\mathcal{A}$ sent through ε-moves during a given execution. We face the second problem thanks to the following property of PABT's: if $\mathcal{A}$ accepts a tree T, it must accept also a tree T' with branching degree at most equal to $d' = n*b(b+1)/2$. This holds since, in each state q and degree g at a node x of the input tree, a set S that satisfies the $\delta(q, g, \mathsf{inp}(x))$ can contain at most $|Q \times \mathbb{N}_{(b)+}|$ pairs of the kind $(\langle g' \rangle, q')$, so we can split each of such a pair in at most g' nodes of degree 1 and then, for each state q', we can have at most $b(b+1)/2$ distinct successors of x. Therefore, it is possible to construct a relative run of $\mathcal{A}'$ by restricting our attention only to trees with degree at most d'. The full construction of $\mathcal{A}'$ and a detailed proof of its correctness is reported in Appendix E. We conclude this proof sketch by only giving some intuition for the transition relation of $\mathcal{A}'$.

Suppose $\mathcal{A} = \langle\{q_0,q_1\},\{a\},2,\delta,q_0,0,F\rangle$, where the δ contains $\delta(q_0,0,a) = (\varepsilon, q_0) \wedge (\langle 2\rangle, q_1)$. Hence, the degree bound d' for $\mathcal{A}$ is 6. Also, suppose that $\mathcal{A}'$ is in the state (H,H'), with $H = \{(q_0,0)\}$ and $H' = \emptyset$. Now, as in the classical case, the set $\{\{(\varepsilon,q_0),(\langle 2\rangle,q_1)\}\}$ satisfies the relation δ for all states in H, but in our construction we can not use this set directly to build δ', since all these tuples contain an additional information (the degree) that we must use to split (H,H') in all possible successors. Accordingly to this fact indeed, we have the following two possibilities: either $\mathcal{A}'$ sends a copy of itself to one child with degree 2 or to two children with degree 1. Moreover, in both cases $\mathcal{A}'$ also sends a witness of the ε-move to direction d'.

Recall that for the NBT $\mathcal{A}'$ with branching degree d' the emptiness problem is solvable in PTIME [VW86] and, precisely, in $O(|Q'|^{2d'})$ (we directly consider the one-letter automaton associated to $\mathcal{A}'$). Then, by Theorem 4, the following result follows.

Corollary 2. *The emptiness problem for a PABT $\mathcal{A}$ with n states and counting branching bound b can be decided in time* $2^{O(n^2 * b^3)}$.

By Theorem 3 and Corollary 2, and since $n = |\mathsf{ecl}(\varphi)| = O(|\varphi|)$ and $b = \mathsf{deg}(\varphi) = O(|\varphi|)$, we get that the satisfiability problem for GCTL is in EXPTIME and precisely solvable in time $2^{O(|\varphi|^5)}$. Since CTL is subsumed by GCTL, the following holds.

Corollary 3. *The satisfiability problem for GCTL is* EXPTIME-COMPLETE.

5 Discussion

Graded modalities refine classical existential and universal quantifiers by specifying the number of elements for which the existential requirement should hold/universal requirement may not hold. Earlier work studied the extension of the μ-calculus by graded modalities and showed that the complexity of the satisfiability problem stays EXPTIME in the graded setting. In this paper, we have introduced and investigated a (semantic) fragment of the graded μ-calculus, that is GCTL, which extends CTL with graded path quantifiers. In particular, we have showed an exponential translation from GCTL to graded μ-calculus, and we claim that there are cases in which this blow-up is unavoidable, making GCTL even more appealing in practice. We are confident on the fact that this result can be achieved by extending to graded automata the automata-theoretic approach used in [Wil99]. A key point of this idea is to show that every GAPT, in order to accept a tree model of a GCTL formula having n grandchildren of the root labeled with the same symbol, needs at least a number of states exponential in n. Since such a tree can be easily accepted by a PABT with a number of states linear in n, we get the desired result of succinctness.

As interesting results about GCTL, we have shown that although this logic is more expressive than CTL, it retains an EXPTIME satisfiability procedure. This result have been achieved by exploiting an automata-theoretic approach via the introduction of a new automata model, that is PATA. As an immediate consequence, by using a classical product automata [KVW00] through PATA's one can also get that the model checking problem is as easy as CTL, i.e. it stays in PTIME. We postpone this to future work. Other directions for future work regard the investigation of graded path modalities along with more complex logics, such as CTL^*, i.e., to investigate GCTL^*. We believe that is not hard to extend to this logic the properties showed for GCTL in Theorem 2 (expressiveness, tree and finite model properties). On the contrary, to evaluate the complexity of the satisfiability problem for GCTL^* is rather than immediate as the automata model we have considered in this paper for GCTL is not appropriate. Indeed, by using a theoretic-automata approach similar to that one used for GCTL, we can reduce the satisfiability problem for GCTL^* to the emptiness problem of PATA, but with an acceptance condition stronger than Büchi, such as the parity one [Mos84]. Unfortunately, the technique we have shown to translate PABT into NBT is not appropriate for parity automata. However, by using a technique based on promises and strategies, as it was done in [KSV02], we conjecture that PATA along with a parity condition can be translated in exponential-time into an alternating parity tree automaton. Then, by using the fact that for the latter the emptiness problem is solvable in exponential-time, we get that the satisfiability problem for GCTL^* is solvable in 2EXPTIME, thus not harder than that for CTL^*. By exploiting a similar idea of that used for graded μ-calculus, one could also show that GCTL^* is equivalent to CTL^* augmented with graded world modalities (Counting-CTL^*[MR03]) and we conjecture that GCTL^* is exponentially more succinct than counting-CTL^*. This result is important as it was shown in [MR03] that Counting-$CTL^*$is equivalent to *monadic path logic*, which is MSOL with set quantifications restricted to paths.

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A Proof of Lemma 1

For all state formulas φ_1 and φ_2 (resp., path formulas ψ_1 and ψ_2), we say that φ_1 *implies* φ_2, formally $\varphi_1 \Rightarrow \varphi_2$, (resp., ψ_1 *implies* ψ_2, formally $\psi_1 \Rightarrow \psi_2$) iff for all Kripke structures $\mathcal{K}$ and worlds $w \in \mathsf{dom}(\mathcal{K})$ it holds that if $\mathcal{K}, w \models \varphi_1$ then $\mathcal{K}, w \models \varphi_2$ (resp., $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \psi_1)) \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \psi_2))$). It is obvious that, φ_1 is *equivalent* to φ_2 (resp., ψ_1 is *equivalent* to ψ_2) iff $\varphi_1 \Rightarrow \varphi_2$ and $\varphi_2 \Rightarrow \varphi_1$ (resp., $\psi_1 \Rightarrow \psi_2$ and $\psi_2 \Rightarrow \psi_1$).

The following two propositions are immediately derived from the semantics of G*CTL**.

Proposition 1. *For all state formulas* φ*, path formulas* ψ*, finite sequences of path formulas* $\{\psi_i\}_i^n$*, and degree* $g \in \mathbb{N}_+$ *it holds that:* (i) $\mathsf{E}^{\geq 0}\psi \equiv \mathfrak{t}$, (ii) $\mathsf{E}^{>g}\psi \Rightarrow \mathsf{E}^{\geq g}\psi$, (iii) $\mathsf{E}\varphi \equiv \varphi$, (iv) $\mathsf{E}^{>g}\varphi \equiv \mathfrak{f}$, (v) $\mathsf{E}^{\geq g}(\varphi \wedge \psi) \equiv \varphi \wedge \mathsf{E}^{\geq g}\psi$, (vi) $\mathsf{E}(\varphi \vee \psi) \equiv \varphi \vee \mathsf{E}\psi$, (vii) $\mathsf{E}^{>g}(\varphi \vee \psi) \equiv \neg\varphi \wedge \mathsf{E}^{>g}\psi$, (viii) $\mathsf{E}\bigwedge_i \psi_i \Rightarrow \bigwedge_i \mathsf{E}\psi_i$, (ix) $\mathsf{E}\bigvee_i \psi_i \equiv \bigvee_i \mathsf{E}\psi_i$, (x) $\mathsf{E}^{>g}\bigvee_i \psi_i \Rightarrow \bigvee_i \mathsf{E}^{<g}\psi_i$, (xi) $\mathsf{E}\tilde{\mathsf{X}}\,\psi \equiv \mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}\mathsf{X}\,\psi$*, and* (xii) $\mathsf{E}^{>g}\tilde{\mathsf{X}}\,\psi \equiv \mathsf{E}^{>g}\mathsf{X}\,\psi \wedge \mathsf{E}\mathsf{X}\,\neg\psi$.

Proposition 2. *For all path formulas* ψ*,* ψ_1*, and* ψ_2*, it holds that:* (i) $\tilde{\mathsf{X}}\,\psi \equiv \tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{X}\,\psi$, (ii) $\psi_1 \mathsf{U}\, \psi_2 \equiv \psi_2 \vee \psi_1 \wedge \mathsf{X}\,(\psi_1 \mathsf{U}\, \psi_2)$*, and* (iii) $\psi_1 \mathsf{R}\, \psi_2 \equiv \psi_2 \wedge (\psi_1 \vee \tilde{\mathsf{X}}\,(\psi_1 \mathsf{R}\, \psi_2))$.

Let X be a set of objects and $\mathrm{R} \subseteq \mathrm{X} \times \mathrm{X}$ be an equivalence relations on X, i.e., R is reflexive, symmetric, and transitive. Then, it is possible to split the set X into a partition of equivalence classes induced by the relation R. Let us denote by $\mathcal{E}_{\mathrm{R}}(\mathrm{X})$ the set of all these equivalence classes, i.e., for all $C_1, C_2 \in \mathcal{E}_{\mathrm{R}}(\mathrm{X})$, with $C_1 \neq C_2$, it holds that *(i)* $\emptyset \neq C_1 \subseteq \mathrm{X}$ and *(ii)* for all elements $x, y \in C_1$ and $z \in C_2$, it holds that $(x, y) \in \mathrm{R}$ and $(x, z) \notin \mathrm{R}$. It is important to remind that for a partition of a set X the following two properties hold: *(i)* $\bigcup_{C \in \mathcal{E}_{\mathrm{R}}(\mathrm{X})} C = \mathrm{X}$ and *(ii)* for all $C_1, C_2 \in \mathcal{E}_{\mathrm{R}}(\mathrm{X})$, with $C_1 \neq C_2$, it holds that $C_1 \cap C_2 = \emptyset$.

Definition 7. **(i-step congruence relation)** *Let* $\mathcal{K}$ *be a Kripke structure and* $\mathfrak{P}$ *be a set of paths in* $\mathsf{paths}(\mathcal{K})$ *such that there is* $i \in \mathbb{N}$ *for which* $\pi \in \mathfrak{P}$ *implies* $|\pi| > i$*. Then, for all paths* $\pi, \pi' \in \mathfrak{P}$*, we say that* π *is* i-step congruent *to* π'*, denoting this by* $\pi \asymp_i \pi'$*, iff for all* $j \in \mathbb{N}_{(i)}$ *it holds that* $\pi(j) = \pi'(j)$*, i.e., the two paths are identical up to the i-th position.*

Definition 8. **(n-size 1-step classes set)** *Let* $\mathcal{E}_{\asymp_1}(\mathfrak{P})$ *be the set of* 1*-step congruence classes on* $\mathfrak{P}$*. Then, by* $I_n(\mathfrak{P})$ *we denote the set of all paths in* $\mathfrak{P}$ *that are in a congruence class of* $\mathfrak{P}$ *itself with cardinality n, i.e.,* $I_n(\mathfrak{P}) = \{\pi \in \mathfrak{P} \mid \exists C \in \mathcal{E}_{\asymp_1}(\mathfrak{P}), |C| = n : \pi \in C\}$.

Lemma 2. *For all finite sets* $\mathfrak{P}$ *it holds that* $\{\frac{|I_n(\mathfrak{P})|}{n}\}_n^{|\mathfrak{P}|} \in \mathsf{P}(|\mathfrak{P}|)$.

Proof. Since $\mathfrak{P}$ is finite, it holds that $\mathcal{E}_{\asymp_1}(\mathfrak{P})$ is finite as well. Consequently, the sets of equivalence classes given by $Q_n = \{C \in \mathcal{E}_{\asymp_1}(\mathfrak{P}) \mid |C| = n\}$ satisfy $|Q_n| < \omega$, i.e., there exists a number $k_n \in \mathbb{N}$ such that $|Q_n| = k_n$. Then, since $I_n(\mathfrak{P}) = \bigcup_{C \in Q_n} C$, it is obvious that $|I_n(\mathfrak{P})| = k_n * n$. By Definition 8, it follows that $\{I_n(\mathfrak{P})\}_n^{|\mathfrak{P}|}$ is a partition of $\mathfrak{P}$, so we have that $\sum_{n=1}^{|\mathfrak{P}|} |I_n(\mathfrak{P})| = |\mathfrak{P}|$, and then $\sum_{n=1}^{|\mathfrak{P}|} n * \frac{|I_n(\mathfrak{P})|}{n} = |\mathfrak{P}|$. Now, by the previous observation, we have that for all numbers $n \in \mathbb{N}$, it holds $\frac{|I_n(\mathfrak{P})|}{n} = k_n \in \mathbb{N}$. Hence, the sequence $\{\frac{|I_n(\mathfrak{P})|}{n}\}_n^{|\mathfrak{P}|}$ is a solution of the Diophantine equation $1 * p_1 + 2 * p_2 + \ldots + |\mathfrak{P}| * p_{|\mathfrak{P}|} = |\mathfrak{P}|$ and thus $\{\frac{|I_n(\mathfrak{P})|}{n}\}_n^{|\mathfrak{P}|} \in \mathsf{P}(|\mathfrak{P}|)$.

Let $\pi \in \mathsf{paths}(\mathcal{K})$ and $n \in \mathbb{N}_{(|\pi|-1)}$. With $\pi_{\geq n}$ we denote the *suffix* of π starting at position n. Formally, *(i)* $|\pi_{\geq n}| = |\pi| - n$ and *(ii)* for all indexes $i \in \mathbb{N}_{(|\pi|-n-1)}$, it holds that $\pi(n+i) = \pi_{\geq n}(i)$.

Lemma 3. *Let $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ be a Kripke structure, $w, w' \in \mathrm{W}$ be two worlds such that $(w, w') \in \mathrm{R}$, and ψ be a GCTL* path formula. Then, it holds that $\mathfrak{P}_A(\mathcal{K}, w', \psi) = \{\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w') \mid \pi \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)\}$.*

Proof. By definition, we have that $\pi \in \mathfrak{P}_A(\mathcal{K}, w', \psi)$ iff for all paths $\pi' \in \mathsf{paths}(\mathcal{K}, w')$ such that $\pi \preccurlyeq \pi'$ it holds that $\mathcal{K}, \pi', 0 \models \psi$. Since $(w, w') \in \mathrm{R}$, for all $\pi, \pi' \in \mathsf{paths}(\mathcal{K}, w')$ there exist $\pi'', \pi''' \in \mathsf{paths}(\mathcal{K}, w)$ such that $\pi = \pi''_{\geq 1}$ and $\pi' = \pi'''_{\geq 1}$, so we have that $\pi \in \mathfrak{P}_A(\mathcal{K}, w', \psi)$ iff for all paths $\pi''' \in \mathsf{paths}(\mathcal{K}, w)$ such that $\pi''_{\geq 1} \preccurlyeq \pi'''_{\geq 1}$ it holds that $\mathcal{K}, \pi'''_{\geq 1}, 0 \models \psi$, thus $\mathcal{K}, \pi''', 1 \models \psi$ and then $\mathcal{K}, \pi''', 0 \models \mathsf{X}\,\psi$. Now, we can observe that, since $\pi'', \pi''' \in \mathsf{paths}(\mathcal{K}, w)$, it holds that $\pi'' \preccurlyeq \pi'''$ iff $\pi''_{\geq 1} \preccurlyeq \pi'''_{\geq 1}$, thus we obtain that $\pi \in \mathfrak{P}_A(\mathcal{K}, w', \psi)$ iff for all paths $\pi''' \in \mathsf{paths}(\mathcal{K}, w)$ such that $\pi'' \preccurlyeq \pi'''$ it holds that $\mathcal{K}, \pi''', 0 \models \mathsf{X}\,\psi$, i.e., $\pi'' \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)$, where $\pi = \pi''_{\geq 1}$. Finally, $\pi \in \mathfrak{P}_A(\mathcal{K}, w', \psi)$ iff $\pi''_{\geq 1} = \pi \in \mathsf{paths}(\mathcal{K}, w')$, with $\pi'' \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)$, i.e., $\pi \in \{\pi''_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w') \mid \pi'' \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)\}$.

Lemma 4. *For all GCTL*path formulas ψ it holds that:*

i) $\mathsf{E}^{\geq g}\mathsf{X}\,\psi \equiv \bigvee_{\{h_i\}_i^g \in \mathsf{CP}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$;

ii) $\mathsf{E}^{\geq g}\tilde{\mathsf{X}}\,\psi \equiv \begin{cases} \mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f} \vee \mathsf{E}\mathsf{X}\,\mathsf{E}\psi, & \text{if } g = 1; \\ \mathsf{E}\mathsf{X}\,\mathsf{E}\neg\psi \wedge \bigvee_{\{h_i\}_i^g \in \mathsf{CP}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi, & \text{otherwise.} \end{cases}$

Proof. Item (i)*, ($\Rightarrow$).* First, assume that $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ is a model for $\mathsf{E}^{\geq g}\mathsf{X}\,\psi$ in $w \in \mathrm{W}$. Then, by definition of the semantic for existential quantifiers, there exists a subset $\mathfrak{P}$ of $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi))$, with $|\mathfrak{P}| = g$. We want to show that, let $h_i = \sum_{n=i}^{g} \frac{|I_n(\mathfrak{P})|}{n}$, it holds $\mathcal{K}, w \models \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$. For each number $n \in \mathbb{N}_{(g)+}$, consider the partition $Q_n = \mathcal{E}_{\asymp_1}(I_n(\mathfrak{P})) = \{C \in \mathcal{E}_{\asymp_1}(\mathfrak{P}) \mid |C| = n\}$ of $I_n(\mathfrak{P})$ in $k_n = \frac{|I_n(\mathfrak{P})|}{n}$ sets. For a fixed $n \in \mathbb{N}_+$, we indicate all these classes with the sequence $\{C_{n,k}\}_k^{k_n}$. Since $C_{n,k} \subseteq \mathfrak{P} \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi))$, it is obvious that all its elements are incomparable minimal paths. Moreover, it is possible to associate a world $w_{n,k}$ to each class $C_{n,k}$ such that for all $\pi \in C_{n,k}$ it holds that $\pi(1) = w_{n,k}$. By Lemma 3, since $(w, w_{n,k}) \in \mathrm{R}$, we have that $\mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi) = \{\pi'_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w_{n,k}) \mid \pi' \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)\}$, so, for all $\pi \in C_{n,k}$, it holds that $\pi_{\geq 1} \in \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi))$. Indeed, $\pi \in \mathfrak{P} \subseteq \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)$ and $\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w_{n,k})$, thus $\pi_{\geq 1} \in \mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi)$. Moreover, since π is minimal in $\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi)$, also $\pi_{\geq 1}$ is minimal in $\mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi)$, because otherwise if there is $\pi' \in \mathsf{paths}(\mathcal{K}, w)$, $\pi' \neq \pi$, such that $\pi'_{\geq 1} \preccurlyeq \pi_{\geq 1}$ we have $\pi' \preccurlyeq \pi$, which contradicts the fact that π is minimal. Now, $|C_{n,k}| = n$ and $\{\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K}, w_{n,k}) \mid \pi \in C_{n,k}\} \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi))$, thus $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w_{n,k}, \psi))| \geq n$. Then, for each $i, n \in \mathbb{N}_{(g)+}$, with $i \leq n$, and for all $k \in \mathbb{N}_{(k_n)+}$, it holds that $\mathcal{K}, w_{n,k} \models \mathsf{E}^{\geq i}\psi$, so for all $\pi \in Q'_n = \{\pi' \in \mathsf{paths}(\mathcal{K}, w) \mid |\pi'| = 2, \exists k \in \mathbb{N}_{(k_n)+} : \pi(1) = w_{n,k}\}$ we have

$\mathcal{K},\pi(1) \models \mathsf{E}^{\geq i}\psi$ that is $\mathcal{K},\pi,1 \models \mathsf{E}^{\geq i}\psi$ and then $\mathcal{K},\pi,0 \models \mathsf{X}\,\mathsf{E}^{\geq i}\psi$. This means that for all $\pi \in \bigcup_{n=i}^{g} Q'_n$ we have $\mathcal{K},\pi,0 \models \mathsf{X}\,\mathsf{E}^{\geq i}\psi$. Observe now that, since each world $w_{n,k}$ is the characteristic world for the equivalence class $C_{n,k} \in \mathcal{E}_{\asymp_1}(\mathfrak{P})$, there is a different world $w_{n,k}$ for each class $C_{n,k}$, so we have that all the sets in $\{Q'_n\}_n^g$ are disjoint and $|Q'_n| = k_n$. It is obvious then that $\bigcup_{n=i}^{g} Q'_n \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq i}\psi))$ so $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq i}\psi))| \geq |\bigcup_{n=i}^{g} Q'_n| = \sum_{n=i}^{g} |Q'_n| = \sum_{n=i}^{g} k_n = \sum_{n=i}^{g} \frac{|I_n(\mathfrak{P})|}{n} = h_i$. Trivially, it follows that $\mathcal{K},w \models \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$ and then $\mathcal{K},w \models \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$. Now, by Lemma 2, we have $\{\frac{|I_n(\mathfrak{P})|}{n}\}_n^g \in \mathsf{P}(g)$, and then, by the definition of the set $\mathsf{CP}(g)$, it holds $\{h_i\}_i^g \in \mathsf{CP}(g)$. Hence, we get the thesis for this direction.

Item (i), *(⇐).* Assume now that $\mathcal{K}$ is a model for $\bigvee_{\{h_i\}_i^g \in \mathsf{CP}(g)} \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$ in $w \in \mathrm{W}$. Then, there is a sequence $\{h_i\}_i^g \in \mathsf{CP}(g)$ such that $\mathcal{K},w \models \bigwedge_{i=1}^{g} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$. Thus, for all indexes $i \in \mathbb{N}_{(g)+}$, it holds that $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq i}\psi))| \geq h_i$, since $\mathcal{K},w \models \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$. Let $k_i = h_i - h_{i+1}$, for $i \in \mathbb{N}_{(g-1)+}$, and $k_g = h_g$. Since $\{h_i\}_i^g \in \mathsf{CP}(g)$, it is obvious that $\{k_i\}_i^g \in \mathsf{P}(g)$. Now, since $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq g}\psi)))| \geq h_g = k_g$, we can construct a set $\mathfrak{P}_g \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq g}\psi))$, with $|\mathfrak{P}_g| = k_g$. Moreover, for all $i \in \mathbb{N}_{(g-1)+}$, let $\mathfrak{P}_i \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq i}\psi)) \setminus \bigcup_{j=i+1}^{g} \mathfrak{P}_j$, with $|\mathfrak{P}_i| = h_i - |\bigcup_{j=i+1}^{g} \mathfrak{P}_j| \leq |(\mathsf{minstructs}(\mathfrak{P}\mathcal{K}w\mathsf{X}\,\mathsf{E}^{\geq i}\psi) \setminus \bigcup_{j=i+1}^{g} \mathfrak{P}_j)|$. It is evident that all the sets $\mathfrak{P}_i$ are disjoint. Furthermore, each of them has just k_i elements. Indeed, by construction we have that $|\mathfrak{P}_g| = k_g$, and, if all sets $\mathfrak{P}_j$, with $j > i$, have cardinality k_i, it holds that $|\mathfrak{P}_i| = h_i - |\bigcup_{j=i+1}^{g} \mathfrak{P}_j| = h_i - \sum_{j=i+1}^{g} |\mathfrak{P}_j| = h_i - \sum_{j=i+1}^{g} k_j = h_i - h_{i+1} = k_i$. Since for all $i \in \mathbb{N}$ it holds that $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq i}\psi)) \supseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq i+1}\psi))$, we have $\mathfrak{P}' = \bigcup_{i=1}^{g} \mathfrak{P}_i \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq 1}\psi))$, so all paths in $\mathfrak{P}'$ are incomparable, i.e. $\mathfrak{P}' = \mathsf{minstructs}(\mathfrak{P}')$. For simplicity, for all $i \in \mathbb{N}_{(g)+}$, we denote with the sequence $\{\pi_{i,j}\}_j^{k_i}$ all the paths into the set $\mathfrak{P}_i$. Note that all paths $\pi_{i,j}$ have length 2. Indeed by definition, $\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq 1}\psi)$ is equal to $\{\pi \in \mathsf{paths}(\mathcal{K},w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K},w) : \pi \preccurlyeq \pi'$ implies $\mathcal{K},\pi',0 \models \mathsf{X}\,\mathsf{E}^{\geq 1}\psi\}$, so, since $\mathcal{K},\pi',0 \models \mathsf{X}\,\mathsf{E}^{\geq 1}\psi$ implies $\mathcal{K},\pi'(1) \models \mathsf{E}^{\geq 1}\psi$ and $\pi(1) = \pi'(1)$, we have $\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq 1}\psi) = \{\pi \in \mathsf{paths}(\mathcal{K},w) \mid \mathcal{K},\pi(1) \models \mathsf{E}^{\geq 1}\psi\}$. Then, applying the minimal structures function to the above sets, we obtain that $\mathfrak{P}' \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\mathsf{E}^{\geq 1}\psi)) = \mathsf{minstructs}(\{\pi \in \mathsf{paths}(\mathcal{K},w) \mid \mathcal{K},\pi(1) \models \mathsf{E}^{\geq 1}\psi\}) = \{\pi \in \mathsf{paths}(\mathcal{K},w) \mid |\pi| = 2,\ \mathcal{K},\pi(1) \models \mathsf{E}^{\geq 1}\psi\}$. Now, for all indexes $i \in \mathbb{N}_{(g)+}$, $j \in \mathbb{N}_{(k_i)+}$, set $w_{i,j} = \pi_{i,j}(1)$. Since all the paths $\pi_{i,j}$ are incomparable paths of length 2 and $\pi_{i,j}(0) = w$, we derive that all the worlds $w_{i,j}$ are different. Moreover, since $\mathcal{K},\pi_{i,j}(1) \models \mathsf{E}^{\geq i}\psi$ it holds also that $\mathcal{K},w_{i,j} \models \mathsf{E}^{\geq i}\psi$ and then $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w_{i,j},\psi))| \geq i$. Thus, since $(w,w_{i,j}) \in \mathrm{R}$, by Lemma 3 we obtain that $|\mathsf{minstructs}(\{\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K},w_{i,j}) \mid \pi \in \mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\psi)\})| \geq i$. At this point, $\pi'_{\geq 1} \in \mathsf{minstructs}(\{\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K},w_{i,j}) \mid \pi \in \mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\psi)\})$ implies that π' is minimal, i.e., $\pi' \in \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\psi))$. Indeed, if this is not the case, there is $\pi'' \in \mathsf{paths}(\mathcal{K},w)$, $\pi'' \neq \pi'$, such that $\pi'' \preccurlyeq \pi'$, and then $\pi''_{\geq 1} \preccurlyeq \pi'_{\geq 1}$ that contradicts the fact that $\pi'_{\geq 1}$ is minimal. Then, let $\mathfrak{P}_{i,j} = \{\pi' \in \mathsf{paths}(\mathcal{K},w) \mid \pi'_{\geq 1} \in \mathsf{minstructs}(\{\pi_{\geq 1} \in \mathsf{paths}(\mathcal{K},w_{i,j}) \mid \pi \in \mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\psi)\})\}$, we have $\mathfrak{P}_{i,j} \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\psi))$. Furtermore, $|\mathfrak{P}_{i,j}| = i$. Let now $\mathfrak{P} = \bigcup_{i=1}^{g} \bigcup_{j=1}^{k_i} \mathfrak{P}_{i,j}$. It is evident that $\mathfrak{P} \subseteq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\psi))$ and then $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\psi))| \geq |\mathfrak{P}|$. Moreover, $|\mathfrak{P}| = \sum_{i=1}^{g} \sum_{j=1}^{k_i} |\mathfrak{P}_{i,j}| = \sum_{i=1}^{g} \sum_{j=1}^{k_i} i = \sum_{i=1}^{g} i * k_i$. Since, as we have previously noted, $\{k_i\}_i^g \in \mathsf{P}(g)$, it holds that

$|\mathfrak{P}| = g$, so $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\,\psi))| \geq g$. The thesis for the other direction follows immediately.

Item (ii). At the formula $\mathsf{E}^{\geq g}\tilde{\mathsf{X}}\,\psi$ we can apply in sequence either the equivalence *(xi)* of Proposition 1, if $g = 1$, or the equivalence *(xii)* of the same proposition and then the item *(i)* of this lemma, obtaining the thesis.

Now, we are able to prove Lemma 1.

Proof. To show the equivalence *(i)*, it is possible to apply, at the formula $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2)$, the following sequence of equivalences: item *(ii)* of Proposition 2, either item *(vi)*, if $g = 1$, or item *(vii)* of Proposition 1 otherwise, item *(v)* of Proposition 1, and, finally, item *(i)* of Lemma 4.

At the same way, to show the equivalence *(ii)*, it is possible to apply, at the formula $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2)$, the following sequence of equivalences: item *(iii)* of Proposition 2, item *(v)* of Proposition 1, either item *(vi)*, if $g = 1$, or item *(vii)* of Proposition 1 otherwise, and, finally, item *(ii)* of Lemma 4.

B Proof of Theorem 1

Proof. Given a GCTL formula φ, we proceed as follows. First we show a fixed point form of the formula derived by the previous equivalences and then we propose a translation which allow us to obtain an equivalent graded μ-calculus formula.

By Lemma 1, we notice that $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2)$ and $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2)$ formulas are definable in a fixed point form. This can be obtained by putting the equivalences written in lemma in terms of functions, that is in a more formal way, we can write $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2) \equiv \mathsf{eu}(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2), \varphi_1, \varphi_2, g)$ and $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2) \equiv \mathsf{er}(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2), \varphi_1, \varphi_2, g)$, for two suitable fixed point functions $\mathsf{eu}(\cdot,\cdot,\cdot,\cdot)$ and $\mathsf{er}(\cdot,\cdot,\cdot,\cdot)$ such that until formulas with degree g do not occur into $\mathsf{eu}(\cdot,\cdot,\cdot,g)$ nor the release ones with degree g into $\mathsf{er}(\cdot,\cdot,\cdot,g)$. For example, when $g > 1$, we have that $\mathsf{eu}(X, \varphi_1, \varphi_2, g) = \neg\varphi_2 \wedge \varphi_1 \wedge \mathsf{ex}'(\varphi_1 \mathsf{U}\, \varphi_2, g) \wedge \mathsf{EX}X$ and $\mathsf{er}(X, \varphi_1, \varphi_2, g) = \varphi_2 \wedge \neg\varphi_1 \wedge \mathsf{EX}\,\mathsf{E}\neg(\varphi_1 \mathsf{R}\, \varphi_2) \wedge \mathsf{ex}'(\varphi_1 \mathsf{R}\, \varphi_2, g) \wedge \mathsf{EX}X$, where $\mathsf{ex}'(\psi, g) = \bigvee_{\{h_i\}_i^g \in \mathsf{CP}'(g)} \bigwedge_{i=1}^{g-1} \mathsf{E}^{\geq h_i}\mathsf{X}\,\mathsf{E}^{\geq i}\psi$, with $\mathsf{CP}'(g) = \mathsf{CP}(g) \setminus \{\{h_i\}_i^g \mid h_g = 1\}$. Note that $|\mathsf{ex}'(\psi, g)| = \Theta((|\psi| + \frac{g}{2}) * 2^{\alpha\sqrt{g}})$, so we have $|\mathsf{eu}(X, \varphi_1, \varphi_2, g)| = |\mathsf{er}(X, \varphi_1, \varphi_2, g)| = \Theta(|\varphi_1| + |\varphi_2| + g * 2^{\alpha\sqrt{g}})$.

Now, w.l.o.g we assume that φ is in existential normal form (we recall that any GCTL formula can be linearly translated in this form). Thanks to the above fixed point functions, we can now conclude the proof by showing a translation function "$\mathsf{trn}(\cdot)$: GCTL $\mapsto$ graded μ-calculus" which allow to get the desired graded μ-calculus formula $\varphi' = \mathsf{trn}(\varphi)$ equivalent to φ. The function $\mathsf{trn}(\cdot)$ is inductively defined as follows: *(i)* $\mathsf{trn}(p) = p$ with $p \in \mathrm{AP}$; *(ii)* $\mathsf{trn}(\neg\varphi) = \neg\,\mathsf{trn}(\varphi)$; *(iii)* $\mathsf{trn}(\mathsf{E}^{\geq 0}\psi) = \mathsf{t}$; *(iv)* $\mathsf{trn}(\mathsf{E}^{\geq g}\mathsf{X}\,\varphi) = \neg\, end \wedge \langle g-1\rangle\, \mathsf{trn}(\varphi)$; *(v)* $\mathsf{trn}(\mathsf{E}^{\geq 1}\tilde{\mathsf{X}}\,\varphi) = end \vee \neg\, end \wedge \langle 0\rangle\, \mathsf{trn}(\varphi)$; *(vi)* $\mathsf{trn}(\mathsf{E}^{> g}\tilde{\mathsf{X}}\,\varphi) = \neg\, end \wedge \langle 0\rangle\, \mathsf{trn}(\neg\varphi) \wedge \langle g\rangle\, \mathsf{trn}(\varphi)$; *(vii)* $\mathsf{trn}(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2)) = \mu X.\mathsf{trn}(\mathsf{eu}(X, \varphi_1, \varphi_2, g))$; *(viii)* $\mathsf{trn}(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2)) = \nu X.\mathsf{trn}(\mathsf{er}(X, \varphi_1, \varphi_2, g))$, where $g \in \mathbb{N}_+$.

By induction on the structure of the formula, it is not hard to see that, for each model $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L}\rangle$ of φ the structure $\mathcal{K}' = \langle \mathrm{AP} \cup \{end\}, \mathrm{W}, \mathrm{R}, \mathrm{L}'\rangle$ is a model of φ', where, let $\mathrm{W}' = \{w \in \mathrm{W} \mid \nexists w' \in \mathrm{W} : (w, w') \in \mathrm{R}\}$, for all $w \in \mathrm{W} \setminus \mathrm{W}'$ and $w' \in \mathrm{W}'$ it holds that $L'(w) = L(w)$ and $L'(w') = L(w') \cup \{end\}$. Moreover, from a model $\mathcal{K} = \langle AP, W, R, L\rangle$ of φ' it is possible to extract one of φ simply substituting the relation R with a new relation R' defined as follows: for all $w, w' \in W$, it holds that $(w, w') \in R'$ iff $(w, w') \in R$ and $end \notin L(w)$.

C Proof of Theorem 2

Consider two Kripke structures $\mathcal{K} = \langle AP, W, R, L\rangle$ and $\mathcal{K}' = \langle AP', W', R', L'\rangle$. We say that $\mathcal{K}$ is *bisimilar* to $\mathcal{K}'$, denoting this by $\mathcal{K} \sim \mathcal{K}'$ iff there exists a non-empty relation $\mathsf{B} \subseteq W \times W'$, called *relation of bisimulation*, such that for all pairs of worlds $(w, w') \in \mathsf{B}$ it holds that: *(i)* $L(w) = L'(w')$; *(ii)* $(w, v) \in R$ implies that there exists a world $v' \in W'$ such that $(v, v') \in \mathsf{B}$ and $(w', v') \in R'$; *(iii)* $(w', v') \in R'$ implies that there exists a world $v \in W$ such that $(v, v') \in \mathsf{B}$ and $(w, v) \in R$. Note that also $\mathsf{B}^{-1} = \{(w', w) \in W' \times W \mid (w, w') \in \mathsf{B}\}$ is a relation of bisimulation.

It is easy to see that an unwinding function is a particular relation of bisimulation.

Proof. Item (i) Let $\mathcal{K} = \langle AP, W, R, L\rangle$ be a Kripke structure. We show that for each GCTL formula φ and world $w \in W$, it holds that $\mathcal{K}, w \models \varphi$ if and only if $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \varphi$. The proof procedes by mutual induction on the structure of the formula φ (external induction) and on the structure of all path formulas it contains (internal induction). Let us start with the external induction. The base step for atomic propositions and the boolean combination cases are easy and left to the reader. Therefore, let us consider the case where φ is of the form $\mathsf{E}^{\geq g}\psi$, for $g \in \mathbb{N}$. The proof proceeds by internal induction on the path formula ψ. As base case, ψ does not contain any quantifier (i.e., ψ is a temporal operators defined on combinations of atomic propositions). First, note that $\mathcal{U}_w^{\mathcal{K}}$ is also an unwinding of itself, so for the construction of $\mathsf{paths}(\mathcal{K}, w)$ and $\mathsf{paths}(\mathcal{U}_w^{\mathcal{K}}, \varepsilon)$ we can choose the same unwinding, obtaining that for all worlds $w \in W$, it holds $\mathsf{paths}(\mathcal{K}, w) = \mathsf{paths}(\mathcal{U}_w^{\mathcal{K}}, \varepsilon)$. Now we show that for all paths $\pi \in \mathsf{paths}(\mathcal{K}, w)$ it holds that $\mathcal{K}, \pi, 0 \models \psi$ if and only if $\mathcal{U}_w^{\mathcal{K}}, \pi, 0 \models \psi$. Indeed, if ψ is a state formula, by the external inductive hypothesis, we obtain the above statement. Then, by induction on the structure of ψ, it is easy to show that the above statement holds for all path formulas. By definition of the satisfiability path set, it follows that $\mathfrak{P}_A(\mathcal{K}, w, \psi) = \mathfrak{P}_A(\mathcal{U}_w^{\mathcal{K}}, \varepsilon, \psi)$. Therefore, by the semantics of the existential quantifiers, we have that $\mathcal{K}, w \models \mathsf{E}^{\geq g}\psi$ if and only if $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{E}^{\geq g}\psi$. Now, let us consider the case where ψ contains $n > 0$ nested quantifiers. For the internal inductive step, we have $\mathcal{K}, w \models \mathsf{E}^{\geq g}\psi'$ if and only if $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{E}^{\geq g}\psi'$, where ψ' contains $n-1$ nested quantifiers. For reasoning analogous to the internal base case, we obtain that $\mathfrak{P}_A(\mathcal{K}, w, \psi) = \mathfrak{P}_A(\mathcal{U}_w^{\mathcal{K}}, \varepsilon, \psi)$, where we recall that ψ contains n nested quantifiers, and then $\mathcal{K}, w \models \mathsf{E}^{\geq g}\psi$ if and only if $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{E}^{\geq g}\psi$. So we have done with the proof.

Item (ii) Consider a GCTL formula φ and suppose that it is satisfiable. Then, there is a model $\mathcal{K}$ for φ in a world $w \in \mathsf{dom}(\mathcal{K})$. By item *(i)*, φ is satisfied at the root of the unwinding $\mathcal{U}_w^{\mathcal{K}}$. Thus, since $\mathcal{U}_w^{\mathcal{K}}$ is a tree, immediately follows that φ is satisfied on a tree model.

Item (iii) Extending, by Lemma 1, the Definition 3.1 of Hintikka structure in [EH85] it is derivable an assertion equivalent to that of Theorem 4.1 in [EH85] itself.

Thus we have that, if φ is satisfiable, it has a "small model", i.e., a model of finite size function of the length of φ.

Item (iv) We show that GCTL distinguishes between bisimilar models. Consider the two trees $\mathcal{T}$ and $\mathcal{T}'$ such as $\mathcal{T}$ contains only the root node and one successor, while $\mathcal{T}'$ contains also another successor. Formally, $\mathcal{T} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$, with $\mathrm{AP} = \emptyset$, $\mathrm{W} = \{\varepsilon, 0\}$, and $\mathrm{R} = \{(\varepsilon, 0)\}$, and $\mathcal{T}' = \langle \mathrm{AP}, \mathrm{W}', \mathrm{R}', \mathrm{L} \rangle$, with $\mathrm{W}' = \mathrm{W} \cup \{1\}$, and $\mathrm{R}' = \mathrm{R} \cup \{(\varepsilon, 1)\}$. From the definition of bisimulation, it immediately follows that $\mathcal{K} \sim \mathcal{K}'$. Now, consider the *GCTL* formula $\varphi = \mathsf{E}^{>1} \mathsf{X}\, \mathfrak{t}$. Then, we have that $\mathfrak{P}_A(\mathcal{T}, \varepsilon, \mathsf{X}\, \mathfrak{t}) = \{\pi\}$, with $\pi(1) = 0$, and $\mathfrak{P}_A(\mathcal{T}', \varepsilon, \mathsf{X}\, \mathfrak{t}) = \{\pi, \pi'\}$, with $\pi'(1) = 1$. Since π and π' are incomparable, it holds that $\{\pi\} = \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{T}, \varepsilon, \mathsf{X}\, \mathfrak{t})) \neq \mathsf{minstructs}(\mathfrak{P}_A(\mathcal{T}', \varepsilon, \mathsf{X}\, \mathfrak{t})) = \{\pi, \pi'\}$, so $\mathcal{T}, \varepsilon \not\models \varphi$, but $\mathcal{T}', \varepsilon \models \varphi$, and then $\mathcal{T} \not\models \varphi$ but $\mathcal{T}' \models \varphi$, i.e., φ is not an invariant on $\mathcal{K}$ and $\mathcal{K}'$.

Item (v) Consider the above two bisimilar tree models $\mathcal{T}$ and $\mathcal{T}'$. Since CTL is invariant under bisimulation, it cannot distinguish between them. Moreover, CTL is a sublogic of GCTL, so we have that the latter can characterize more models than those characterizable by the former logic. Then, it follows that GCTL is more expressive than CTL.

D Construction and correctness proof of Theorem 3

First, we recall the classical definition of *Fischer-Ladner closure* $\mathsf{cl}(\varphi)$ of φ, i.e., the set of all state formulas contained in φ (including φ). Let $g \in \mathbb{N}_+$, $\mathtt{Qnt} \in \{\mathsf{E}^{\geq g}, \mathsf{A}^{<g}\}$, $\mathtt{Op} \in \{\wedge, \vee\}$, $\mathtt{Op}' \in \{\mathsf{X}, \tilde{\mathsf{X}}\}$ and $\mathtt{Op}'' \in \{\mathsf{U}, \mathsf{R}\}$ we have: *(i)* $\varphi \in \mathsf{cl}(\varphi)$, *(ii)* if $\varphi_1 \mathtt{Op}\, \varphi_2 \in \mathsf{cl}(\varphi)$ then $\varphi_1, \varphi_2 \in \mathsf{cl}(\varphi)$, *(iii)* if $\mathtt{Qnt}\, \mathtt{Op}' \varphi' \in \mathsf{cl}(\varphi)$ then $\varphi' \in \mathsf{cl}(\varphi)$, and *(iv)* if $\mathtt{Qnt}\, (\varphi_1 \mathtt{Op}'' \varphi_2) \in \mathsf{cl}(\varphi)$ then $\varphi_1, \varphi_2 \in \mathsf{cl}(\varphi)$. Let $\natural\varphi'$ denote the GCTL formula in positive normal form equivalent to $\neg\varphi'$. The extended closure $\mathsf{ecl}(\varphi)$ satisfies all the above properties of $\mathsf{cl}(\varphi)$ and additionally it satisfies the following: for all $g \in \mathbb{N}_+$, $\mathtt{Op} \in \{\mathsf{X}, \tilde{\mathsf{X}}\}$, and ψ until or release GCTL path formula, it holds that *(i)* if $\mathsf{E}^{\geq g} \mathtt{Op}\, \varphi' \in \mathsf{ecl}(\varphi)$ then $\langle \varphi' \rangle, \langle \natural\varphi' \rangle \in \mathsf{ecl}(\varphi)$, *(ii)* if $\mathsf{A}^{<g} \mathtt{Op}\, \varphi' \in \mathsf{ecl}(\varphi)$ then $[\varphi'], [\natural\varphi'] \in \mathsf{ecl}(\varphi)$, *(iii)* if $\mathsf{E}^{\geq g} \psi \in \mathsf{ecl}(\varphi)$ then $\langle \psi \rangle, \langle \natural\psi \rangle \in \mathsf{ecl}(\varphi)$, *(iv)* if $\mathsf{A}^{<g} \psi \in \mathsf{ecl}(\varphi)$ then $[\psi], [\natural\psi] \in \mathsf{ecl}(\varphi)$, *(v)* if $\langle \varphi_1 \mathsf{U}\, \varphi_2 \rangle$ or $[\varphi_1 \mathsf{R}\, \varphi_2]$ are in $\mathsf{ecl}(\varphi)$ then $\natural\varphi_2 \in \mathsf{ecl}(\varphi)$, and *(iv)* if $\langle \varphi_1 \mathsf{R}\, \varphi_2 \rangle$ or $[\varphi_1 \mathsf{U}\, \varphi_2]$ are in $\mathsf{ecl}(\varphi)$ then $\natural\varphi_1 \in \mathsf{ecl}(\varphi)$. It is obvious that $|\mathsf{ecl}(\varphi)| = O(|\varphi|)$.

The formal definition of the δ follows. For all $\sigma \in 2^{\mathrm{AP}}$ and $g, h \in \mathbb{N}_{(b)+}$, with $h \neq 1$, we set:

- $\delta(\mathfrak{t}, 0, \sigma) = \mathfrak{t}$
- $\delta(\mathfrak{f}, 0, \sigma) = \mathfrak{f}$
- $\delta(p, 0, \sigma) = (p \in \sigma)$
- $\delta(\neg p, 0, \sigma) = (p \notin \sigma)$
- $\delta(\varphi_1 \wedge \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \wedge (\varepsilon, \varphi_2)$
- $\delta(\varphi_1 \vee \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \vee (\varepsilon, \varphi_2)$
- $\delta(\mathsf{E}^{\geq g} \mathsf{X}\, \varphi, 0, \sigma) = (\langle g \rangle, \langle \varphi \rangle)$
- $\delta(\mathsf{A}^{<g} \tilde{\mathsf{X}}\, \varphi, 0, \sigma) = ([g], [\varphi])$
- $\delta(\mathsf{E} \tilde{\mathsf{X}}\, \varphi, 0, \sigma) = ([1], \mathfrak{f}) \vee (\langle 1 \rangle, \langle \varphi \rangle)$
- $\delta(\mathsf{A} \mathsf{X}\, \varphi, 0, \sigma) = (\langle 1 \rangle, \mathfrak{t}) \wedge ([1], [\varphi])$
- $\delta(\mathsf{E}^{\geq h} \tilde{\mathsf{X}}\, \varphi, 0, \sigma) = (\langle 1 \rangle, \langle \natural\varphi \rangle) \wedge (\langle h \rangle, \langle \varphi \rangle)$
- $\delta(\mathsf{A}^{<h} \mathsf{X}\, \varphi, 0, \sigma) = ([1], [\natural\varphi]) \vee ([h], [\varphi])$
- $\delta(\langle \varphi \rangle, 1, \sigma) = (\varepsilon, \varphi)$
- $\delta([\varphi], 1, \sigma) = (\varepsilon, \varphi)$
- $\delta(\langle \varphi \rangle, h, \sigma) = \mathfrak{f}$
- $\delta([\varphi], h, \sigma) = \mathfrak{t}$
- $\delta(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{U}\, \varphi_2), 0, \sigma) = \delta(\langle \varphi_1 \mathsf{U}\, \varphi_2 \rangle, g, \sigma)$
- $\delta(\mathsf{A}^{<g}(\varphi_1 \mathsf{U}\, \varphi_2), 0, \sigma) = \delta([\varphi_1 \mathsf{U}\, \varphi_2], g, \sigma)$
- $\delta(\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R}\, \varphi_2), 0, \sigma) = \delta(\langle \varphi_1 \mathsf{R}\, \varphi_2 \rangle, g, \sigma)$
- $\delta(\mathsf{A}^{<g}(\varphi_1 \mathsf{R}\, \varphi_2), 0, \sigma) = \delta([\varphi_1 \mathsf{R}\, \varphi_2], g, \sigma)$

- $\delta(\langle \varphi_1 \mathsf{U} \varphi_2 \rangle, 1, \sigma) = (\varepsilon, \varphi_2) \vee (\varepsilon, \varphi_1) \wedge (\langle 1 \rangle, \langle \varphi_1 \mathsf{U} \varphi_2 \rangle)$
- $\delta(\langle \varphi_1 \mathsf{U} \varphi_2 \rangle, h, \sigma) = (\varepsilon, \natural\varphi_2) \wedge (\varepsilon, \varphi_1) \wedge (\langle h \rangle, \langle \varphi_1 \mathsf{U} \varphi_2 \rangle)$
- $\delta([\varphi_1 \mathsf{U} \varphi_2], 1, \sigma) = (\varepsilon, \varphi_2) \vee (\varepsilon, \varphi_1) \wedge (\langle 1 \rangle, \mathfrak{t}) \wedge ([1], [\varphi_1 \mathsf{U} \varphi_2])$
- $\delta([\varphi_1 \mathsf{U} \varphi_2], h, \sigma) = (\varepsilon, \varphi_2) \vee (\varepsilon, \natural\varphi_1) \vee ([1], [\natural(\varphi_1 \mathsf{U} \varphi_2)]) \vee ([h], [\varphi_1 \mathsf{U} \varphi_2])$
- $\delta(\langle \varphi_1 \mathsf{R} \varphi_2 \rangle, 1, \sigma) = (\varepsilon, \varphi_2) \wedge ((\varepsilon, \varphi_1) \vee ([1], \mathfrak{f}) \vee (\langle 1 \rangle, \langle \varphi_1 \mathsf{R} \varphi_2 \rangle))$
- $\delta(\langle \varphi_1 \mathsf{R} \varphi_2 \rangle, h, \sigma) = (\varepsilon, \varphi_2) \wedge (\varepsilon, \natural\varphi_1) \wedge (\langle 1 \rangle, \langle \natural(\varphi_1 \mathsf{R} \varphi_2) \rangle) \wedge (\langle h \rangle, \langle \varphi_1 \mathsf{R} \varphi_2 \rangle)$
- $\delta([\varphi_1 \mathsf{R} \varphi_2], 1, \sigma) = (\varepsilon, \varphi_2) \wedge ((\varepsilon, \varphi_1) \vee ([1], [\varphi_1 \mathsf{R} \varphi_2]))$
- $\delta([\varphi_1 \mathsf{R} \varphi_2], h, \sigma) = (\varepsilon, \natural\varphi_2) \vee (\varepsilon, \varphi_1) \vee ([h], [\varphi_1 \mathsf{R} \varphi_2])$

Lemma 5. *For all sequences $\{h_i\}_i^g \in \mathsf{CP}(g)$ and $\{h'_i\}_i^{g-1} \in \mathsf{CP}(g-1)$, with $h_g = 0$, there exists an index $j \in \mathbb{N}_{(g-1)+}$ such that $h'_j < h_j$.*

Proof. If for contradiction for all $j \in \mathbb{N}_{(g-1)+}$ we have $h'_j \geq h_j$ then, since $h_g = 0$, we would find out that $g - 1 = \sum_{j=1}^{g-1} h'_j \geq \sum_{j=1}^{g-1} h_j = \sum_{j=1}^{g} h_j = g$, that is impossible.

Lemma 6. *For all sequences $\{M_i\}_i^{g+1} \in \mathsf{spart}(D, [g])$ it holds that $|M_g| \leq 1$.*

Proof. If $g = 2$, there are only two sequences $\{h_i\}_i^2 \in \mathsf{CP}(2)$ and those are $h_1 = 1$ and $h_2 = 1$ or $h_1 = 2$ and $h_2 = 0$. Since in both the cases we have $|M_2| < h_1$ it holds that $|M_2| < 2$. Consider now $g > 2$. Then, in $\mathsf{CP}(g)$ there exists the sequence $\{h_i\}_i^2$ such that $h_1 = 2$, $h_2 = \ldots = h_{g-1} = 1$, and $h_g = 0$. By Definition 4, there exists a j such that $|M_{j+1}| < h_j$. It cannot be $j = g$ since $|M_{g+1}| < h_g = 0$ is false. If $j = g - 1$ it holds $|M_g| < h_{g-1} = 1$ and then we have done with the proof. Finally, for $j < g - 1$, it holds $|M_{j+1}| < h_j \leq 2$. It is known that $M_g \subseteq \ldots \subseteq M_{j+1}$ so we have $|M_g| \leq |M_{j+1}| < 2$.

Lemma 7. *Let $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ be a Kripke structure and $w \in \mathrm{W}$ be a world. Then, $\mathcal{K}, w \models \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$ (resp., $\mathcal{K}, w \models \mathsf{A}^{<1} \mathsf{X}\, \mathfrak{t}$) iff $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) = \emptyset$, (resp., $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \neq \emptyset$).*

Proof. Then, $\mathcal{K}, w \models \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$ iff $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \tilde{\mathsf{X}}\, \mathfrak{f})) \neq \emptyset$, that is $\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi'$ implies $\mathcal{K}, \pi', 0 \models \tilde{\mathsf{X}}\, \mathfrak{f}\} \neq \emptyset$. Now, since for all $\pi \in \mathsf{paths}(\mathcal{K}, w)$ we have $\mathcal{K}, \pi, 0 \models \tilde{\mathsf{X}}\, \mathfrak{f}$ iff $|\pi| = 1$, it holds that $\mathcal{K}, w \models \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$ iff $\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi'$ implies $|\pi| = 1\} \neq \emptyset$, that is $\mathcal{K}, w \models \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$ iff for all $\pi \in \mathsf{paths}(\mathcal{K}, w)$ it holds $|\pi| = 1$, and then there is no world $w' \in \mathsf{dom}(\mathcal{K})$ such that $(w, w') \in \mathrm{R}$, hence $\mathcal{K}, w \models \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$ iff $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) = \emptyset$. Now, since $\mathcal{K}, w \models \mathsf{A}^{<1} \mathsf{X}\, \mathfrak{t}$ iff $\mathcal{K}, w \models \neg \mathsf{E}^{\geq 1} \tilde{\mathsf{X}}\, \mathfrak{f}$, we have that $\mathcal{K}, w \models \mathsf{A}^{<1} \mathsf{X}\, \mathfrak{t}$ iff $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \neq \emptyset$.

Lemma 8. *Let $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ be a Kripke structure, $w \in \mathrm{W}$ be a world, and φ be a GCTL state formula. Then, it holds that (i) $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\, \varphi))| = |\{w' \in \mathrm{W} \mid (w, w') \in \mathrm{R},\ \mathcal{K}, w' \models \varphi\}|$ and (ii) $|\mathsf{minstructs}(\mathsf{paths}(\mathcal{K}, w) \setminus \mathfrak{P}_E(\mathcal{K}, w, \tilde{\mathsf{X}}\, \varphi))| = |\{w' \in \mathrm{W} \mid (w, w') \in \mathrm{R},\ \mathcal{K}, w' \models \neg\varphi\}|$.*

Proof. Now, we prove the equality *(i)*. The equality *(ii)* easily follows by the duality equality $\mathfrak{P}_A(\mathcal{K}, w, \neg\varphi) = \mathsf{paths}(\mathcal{K}, w) \setminus \mathfrak{P}_E(\mathcal{K}, w, \varphi)$.

By definition, it holds that $\mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\, \varphi)$ is equal to $\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi'$ implies $\mathcal{K}, \pi', 0 \models \mathsf{X}\, \varphi\}$, so it is equal to $\{\pi \in \mathsf{paths}(\mathcal{K}, w) \mid \forall \pi' \in \mathsf{paths}(\mathcal{K}, w) : \pi \preccurlyeq \pi'$ implies $\mathcal{K}, \pi', 1 \models \varphi\}$. It is evident then that for each path $\pi \in \mathfrak{P}_A(\mathcal{K}, w, \mathsf{X}\, \varphi)$ it holds that $\mathcal{K}, \pi, 1 \models \varphi$, so, since φ is a state formula, we have that

$\mathcal{K},\pi(1) \models \varphi$ and then $\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\varphi) = \{\pi \in \mathsf{paths}(\mathcal{K},w) \mid |\pi| > 1,\ \mathcal{K},\pi(1) \models \varphi\}$. Now, it is obvious that minimal paths in the set $\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\varphi)$ are all the paths of length 2, starting in w, and which satisfy φ on the second world, so we obtain that the set $\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{K},w,\mathsf{X}\,\varphi))$ is equal to $\{\pi \in \mathsf{paths}(\mathcal{K},w) \mid |\pi| = 2,\ \mathcal{K},\pi(1) \models \varphi\}$. Since the paths of length two, which have w as their first world, are as many as the successors of w itself, because such paths are made by w and by one of its successors that is $(\pi(0),\pi(1)) = (w,\pi(1)) \in \mathrm{R}$, we have that $|\{\pi \in \mathsf{paths}(\mathcal{K},w) \mid |\pi| = 2,\ \mathcal{K},\pi(1) \models \varphi\}|$ is equal to $|\{w' \in \mathrm{W} \mid (w,w') \in \mathrm{R},\ \mathcal{K},w' \models \varphi\}|$, thus the thesis follows.

Definition 9. (Partial run of a PABT) *A* partial run *of a PABT* $\mathcal{A} = \langle Q,\Sigma,b,\delta,q_0,g_0,F\rangle$ *on a Σ-labeled tree* $\langle \mathsf{T},\mathsf{inp}\rangle$ *is a* $(\mathsf{T}\times Q\times \mathbb{N}_{(b)}\times \mathbb{N}_{(1)})$*-labeled full tree* $\langle \mathsf{T}_{\mathsf{pr}},\mathsf{prun}\rangle$ *satisfying the following conditions:*

1. $\mathsf{prun}(\varepsilon) = (\varepsilon,q_0,g_0,l_0)$*, for some* $l_0 \in \mathbb{N}_{(1)}$*;*
2. *for all* $y \in \mathsf{T}_{\mathsf{pr}}$ *with* $\mathsf{prun}(y) = (x,q,g,0)$*, it holds that* $\mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(y) = \emptyset$*, i.e., y have no successors;*
3. *for all* $y \in \mathsf{T}_{\mathsf{pr}}$ *with* $\mathsf{prun}(y) = (x,q,g,1)$*, there exists a set* $S \subseteq D_b^{\varepsilon} \times Q$*, where* $S \models \delta(q,g,\mathsf{inp}(x))$*, and a set* $E \in \mathsf{exec}(S,\mathsf{dir}_{\mathsf{T}}(x))$ *such that for all configurations* $(d,q',g') \in E$ *there is a node* $y' \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(y)$ *such that* $\mathsf{prun}(y') = (x\cdot d,q',g',l)$*, for some* $l \in \mathbb{N}_{(1)}$*.*

A 0*-labeled (resp.,* 1*-labeled) node is a node with label that ends with* 0 *(resp.,* 1*). The partial run with all* 1*-labeled nodes is called a* 1*-labeled partial run. Finally, the partial run* $\langle \mathsf{T}_{\mathsf{pr}},\mathsf{prun}\rangle$ *is* accepting *iff all its infinite paths satisfy the acceptance condition, i.e., for all paths* $\pi \preccurlyeq \mathsf{T}_{\mathsf{pr}}$*, with* $|\pi| = \omega$*, it holds that* $\mathsf{inf}(\mathsf{prun}_{|\pi}) \cap \mathsf{T}\times F\times \mathbb{N}_{(1)} \neq \emptyset$*.*

It's evident that, the projection of a 1-labeled partial run on $\mathsf{T}\times Q\times \mathbb{N}_{(b)}$ is a run, moreover, if such a partial run is accepting the corresponding run is accepting too. In addition, if there exists a run, we can build a corresponding partial run by adding to all labels a 1 at the end of the labels.

Let $a \in \mathbb{N}^*$ and $\mathrm{X} \subseteq \mathbb{N}^*$. Then, with $a \triangleleft \mathrm{X}$ and $a \triangleright \mathrm{X}$ we denote, respectively, the two sets $\{x \in \mathbb{N}^* \mid a\cdot x \in \mathrm{X}\}$ and $\{a\cdot x \in \mathbb{N}^* \mid x \in \mathrm{X}\}$. Moreover, let $\langle \mathrm{X},\mathrm{L}\rangle$ be a labeled tree. Then, with $a \triangleleft \langle \mathrm{X},\mathrm{L}\rangle$ we denote the labeled tree $\langle \mathrm{X}',\mathrm{L}'\rangle$, where it is set $\mathrm{X}' = a \triangleleft \mathrm{X}$ and, for all $x \in \mathrm{X}'$, $\mathrm{L}'(x) = \mathrm{L}(a\cdot x)$.

Definition 10. (Extention of a partial run) *Let us consider a partial run* $\langle \mathsf{T}_{\mathsf{pr}},\mathsf{prun}\rangle$ *of a PABT* $\mathcal{A} = \langle Q,\Sigma,b,\delta,q_0,g_0,F\rangle$ *on an input* $\langle \mathsf{T},\mathsf{inp}\rangle$*, with a sequence of nodes* $\{y_i\}_i^n \subseteq \mathsf{T}_{\mathsf{pr}}$ *such that* $\mathsf{prun}(y_i) = (x_i,q_i,g_i,0)$*, for all indexes* $i \in \mathbb{N}_{(n)+}$*. Consider also a sequence of partial runs* $\{\langle \mathsf{T}_{\mathsf{pr}_i},\mathsf{prun}_i\rangle\}_i^n$ *of a sequence of PABTs* $\{\mathcal{A}_i\}_i^n$*,* $\mathcal{A}_i = \langle Q,\Sigma,b,\delta,q_i,g_i,F\rangle$*, on the inputs* $\{x_i \triangleleft \langle \mathsf{T},\mathsf{inp}\rangle\}_i^n$*. Then, we call an extension of* $\langle \mathsf{T}_{\mathsf{pr}},\mathsf{prun}\rangle$ *with respect to* $\{\langle \mathsf{T}_{\mathsf{pr}_i},\mathsf{prun}_i\rangle\}_i^n$ *on the nodes* $\{y_i\}_i^n$*, a* $(\mathsf{T}\times Q\times \mathbb{N}_{(b)}\times \mathbb{N}_{(1)})$*-labeled tree* $\langle {\mathsf{T}_{\mathsf{pr}}}',\mathsf{prun}'\rangle$ *obtained by substituting each node* y_i *with the tree* $\langle \mathsf{T}_{\mathsf{pr}_i},\mathsf{prun}_i\rangle$*. More formally, we construct* $\langle {\mathsf{T}_{\mathsf{pr}}}',\mathsf{prun}'\rangle$ *as follows:* (i) ${\mathsf{T}_{\mathsf{pr}}}' = \mathsf{T}_{\mathsf{pr}} \cup \bigcup_{i=1}^n (y_i \triangleright \mathsf{T}_{\mathsf{pr}_i})$*;* (ii) *for all* $z \in \mathsf{T}_{\mathsf{pr}} \setminus \bigcup_{i=1}^n \{y_i\}$ *it holds that* $\mathsf{prun}'(z) = \mathsf{prun}(z)$*;* (iii) *for all* $i \in \mathbb{N}_{(n)+}$ *and* $z \in {\mathsf{T}_{\mathsf{pr}}}'$*, with* $z = y_i\cdot y'$ *and* $\mathsf{prun}_i(y') = (x',q',g',l')$*, it holds that* $\mathsf{prun}'(z) = (x_i\cdot x',q',g',l')$*.*

Lemma 9. *Let* $\langle \mathsf{T}_{\mathsf{pr}},\mathsf{prun}\rangle$ *be a partial run of a PABT* $\mathcal{A} = \langle Q,\Sigma,b,\delta,q_0,g_0,F\rangle$ *on an input* $\langle \mathsf{T},\mathsf{inp}\rangle$*, with a sequence of nodes* $\{y_i\}_i^n \subseteq \mathsf{T}_{\mathsf{pr}}$ *such that* $\mathsf{prun}(y_i) = (x_i,q_i,g_i,$

0) *for all indexes* $i \in \mathbb{N}_{(n)+}$, *and* $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^n$ *be a a sequence of partial run of a sequence of PABTs* $\{\mathcal{A}_i\}_i^n$, $\mathcal{A}_i = \langle Q, \Sigma, b, \delta, q_i, g_i, F\rangle$, *on the inputs* $\{\langle \mathsf{T}_i, \mathsf{inp}_i \rangle\}_i^n$, *where* $\langle \mathsf{T}_i, \mathsf{inp}_i \rangle = x_i \triangleleft \langle \mathsf{T}, \mathsf{inp} \rangle$. *Then, we have that:*

1. *the extension* $\langle \mathsf{T}_{\mathsf{pr}}{}', \mathsf{prun}' \rangle$ *of* $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ *with respect to* $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^n$ *on the nodes* $\{y_i\}_i^n$ *is a partial run of* $\mathcal{A}$ *on* $\langle \mathsf{T}, \mathsf{inp} \rangle$*;*
2. *if the partial runs* $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ *and* $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^n$ *are accepting then* $\langle \mathsf{T}_{\mathsf{pr}}{}', \mathsf{prun}' \rangle$ *is accepting as well;*
3. *if* $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^n$ *are* 1*-labeled and* $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ *has only* $\{y_i\}_i^n$ *as* 0*-labeled nodes then* $\langle \mathsf{T}_{\mathsf{pr}}{}', \mathsf{prun}' \rangle$ *is* 1*-labeled as well.*

Proof. Item (i) We show that the three properties in Definition 9 of partial run holds.

1. Property of the root.
   (a) If it is labeled by $(\varepsilon, q, h, 1)$ then its label in $\langle \mathsf{T}_{\mathsf{pr}}{}', \mathsf{prun}' \rangle$ remains the same.
   (b) If it is labeled by $(\varepsilon, q, h, 0)$ it has no successor, so there exists a unique node $y_1 \in \mathsf{T}_{\mathsf{pr}}$ such that $\mathsf{prun}(y_1) = (x_1, q_1, g_1, 0)$. The corresponding partial run $\langle \mathsf{T}_{\mathsf{pr}\,1}, \mathsf{prun}_1 \rangle$ has a root labeled by (ε, q, h, l), with $l \in\in \mathbb{N}_{(1)}$, thus, by substitution, the root has the same label in $\langle \mathsf{T}_{\mathsf{pr}}{}', \mathsf{prun}' \rangle$.
2. Property of 0-labeled nodes.
   (a) For all $z \in \mathsf{T}_{\mathsf{pr}}{}'$, such that for all $i \in \mathbb{N}_{(n)+}$ it holds that y_i is not a prefix of z, we have that $\mathsf{prun}'(z) = \mathsf{prun}(z) = (x, q, g, 0)$. Moreover, since $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is a partial run, z has no successor in it, so it holds that $\mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}{}'}(z) = \emptyset$.
   (b) For all $z \in \mathsf{T}_{\mathsf{pr}}{}'$, such that there exists $i \in \mathbb{N}_{(n)+}$ for which it holds that $z = y_i \cdot y$, we have that $\mathsf{prun}'(z) = (x_i \cdot x, q, g, 0)$, since $\mathsf{prun}_i(y) = (x, q, g, 0)$. Moreover, since $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ is a partial run, y has no successor in it, so it holds that $\mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}{}'}(z) = \emptyset$.
3. Property of 1-labeled nodes.
   (a) For all $z \in \mathsf{T}_{\mathsf{pr}}{}'$, such that for all $i \in \mathbb{N}_{(n)+}$ it holds that y_i is not a prefix of z, we have that $\mathsf{prun}'(z) = \mathsf{prun}(z) = (x, q, g, 1)$. Moreover, for all successors $z' \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}{}'}(z)$ with $\mathsf{prun}(z') = (x', q', h', l')$, it holds that $\mathsf{prun}'(z') = (x', q', h', l'')$, where l' may be not equal to l'' only if $z' = y$. Now, since the property expressed in item 3 only depends on the first three components of the labels and $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is a partial run, it necessarily holds that item 3 also holds between z and its successors in $\langle \mathsf{T}_{\mathsf{pr}}{}', \mathsf{prun}' \rangle$.
   (b) For all $z \in \mathsf{T}_{\mathsf{pr}}{}'$, such that there exists $i \in \mathbb{N}_{(n)+}$ for which it holds that $z = y_i \cdot y$ with $\mathsf{prun}'(z) = (x_i \cdot x, q, g, 1)$ and $\mathsf{prun}_i(y) = (x, q, g, 1)$, we have that, since $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ is a partial run, there exists a set $S \subseteq D_b^{\varepsilon} \times Q$, where $S \models \delta(q, g, \mathsf{inp}_i(x))$, and a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathsf{T}_i}(x))$ such that for all configurations $(d, q', g') \in E$ there is a node $y' \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}_i}}(y)$ such that $\mathsf{prun}_i(y') = (x \cdot d, q', g', l)$. Now, since $\mathsf{inp}'(x_i \cdot x) = \mathsf{inp}_i(x)$ and $\mathsf{dir}_{\mathsf{T}'}(x_i \cdot x) = \mathsf{dir}_{\mathsf{T}_i}(x)$ we have that there exists a set $S \subseteq D_b^{\varepsilon} \times Q$, where $S \models \delta(q, g, \mathsf{inp}'(x_i \cdot x))$, and a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathsf{T}'}(x_i \cdot x))$ such that for all configurations $(d, q', g') \in E$ there is a node $z' = y_i \cdot y' \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}{}'}(z)$ such that $\mathsf{prun}'(z') = (x_i \cdot x \cdot d, q', g', l)$.

Item (ii) Let us consider an infinite path $\pi \preccurlyeq \mathsf{T}_{\mathsf{pr}}{}'$. Then, two situations can arise.

1. If $\pi \preccurlyeq \mathsf{T}_{\mathsf{pr}}$, we have that $\mathsf{inf}(\mathsf{prun}'_{|\pi}) \cap \mathsf{T} \times F \neq \emptyset$, since $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is accepting and $\mathsf{prun}'_{|\pi} = \mathsf{prun}_{|\pi}$.
2. If $\pi \not\preccurlyeq \mathsf{T}_{\mathsf{pr}}$ then there exists an index $i \in \mathbb{N}_{(n)+}$ and a path $\pi' \preccurlyeq \mathsf{T}_{\mathsf{pr}\,i}$ such that $\pi_{\geq |y_i|} = \pi'$. Since $\langle \mathsf{T}_{\mathsf{pr}\,i}, \mathsf{prun}_i \rangle$ is accepting and $\mathsf{inf}(\mathsf{prun}'_{|\pi}) = \mathsf{inf}(\mathsf{prun}_{i|\pi'})$, we have that $\mathsf{inf}(\mathsf{prun}'_{|\pi}) \cap \mathsf{T} \times F \neq \emptyset$.

Item (iii) Finally, let us consider a node $z \in \mathsf{T}_{\mathsf{pr}}{}'$. Then, two situations can arise.

1. If $z \in \mathsf{T}_{\mathsf{pr}}$ and $z \neq y_i$, for all indexes $i \in \mathbb{N}_{(n)+}$, it holds that $\mathsf{prun}'(z) = \mathsf{prun}(z)$, so z is 1-labeled.
2. If there is an index $i \in \mathbb{N}_{(n)+}$ such that $z = y_i \cdot y$, it holds that $\mathsf{prun}'(z) = (x_i \cdot x, q, g, 1)$, since $\mathsf{prun}_i(y) = (x, q, g, 1)$, so z is also 1-labeled.

Definition 11. (Extraction of a partial run) *Let us consider a partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ of a PABT $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$ on an input $\langle \mathsf{T}, \mathsf{inp} \rangle$, with a sequence of nodes $\{y_i\}_i^n \subseteq \mathsf{T}_{\mathsf{pr}}$ such that $\mathsf{prun}(y_i) = (x_i, q_i, g_i, l_i)$ for all indexes $i \in \mathbb{N}_{(n)+}$. Then, we call an extraction of $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ on the nodes $\{y_i\}_i^n$ a sequence of $(\mathsf{T} \times Q \times \mathbb{N}_{(b)} \times \mathbb{N}_{(1)})$-labeled trees $\{\langle \mathsf{T}_{\mathsf{pr}\,i}, \mathsf{prun}_i \rangle\}_i^n$, where $\langle \mathsf{T}_{\mathsf{pr}\,i}, \mathsf{prun}_i \rangle$ is given by the subtree rooted at node y_i for all indexes $i \in \mathbb{N}_{(n)+}$. More formally, we construct a $\langle \mathsf{T}_{\mathsf{pr}\,i}, \mathsf{prun}_i \rangle$ as follows:* (i) $\mathsf{T}_{\mathsf{pr}\,i} = y_i \triangleleft \mathsf{T}_{\mathsf{pr}}$; (ii) *for all $z \in \mathsf{T}_{\mathsf{pr}\,i}$, with $\mathsf{prun}(y_i \cdot z) = (x_i \cdot x', q', g', l')$, it holds that $\mathsf{prun}_i(z) = (x', q', g', l')$.*

Lemma 10. *Let $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ be a partial run of a PABT $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$ on an input $\langle \mathsf{T}, \mathsf{inp} \rangle$, with a sequence of nodes $\{y_i\}_i^n \subseteq \mathsf{T}_{\mathsf{pr}}$ such that $\mathsf{prun}(y_i) = (x_i, q_i, g_i, l_i)$ for all indexes $i \in \mathbb{N}_{(n)+}$. Then we have:*

1. *the extraction $\{\langle \mathsf{T}_{\mathsf{pr}\,i}, \mathsf{prun}_i \rangle\}_i^n$ of $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ on the nodes $\{y_i\}_i^n$ is a sequence of partial runs of the sequence of PABTs $\{\mathcal{A}_i\}_i^n$, $\mathcal{A}_i = \langle Q, \Sigma, b, \delta, q_i, g_i, F \rangle$, on the inputs $\{\langle \mathsf{T}_i, \mathsf{inp}_i \rangle\}_i^n$, where $\langle \mathsf{T}_i, \mathsf{inp}_i \rangle = x_i \triangleleft \langle \mathsf{T}, \mathsf{inp} \rangle$;*
2. *if the partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is accepting then all the partial runs $\{\langle \mathsf{T}_{\mathsf{pr}\,i}, \mathsf{prun}_i \rangle\}_i^n$ are accepting as well;*
3. *if $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is 1-labeled then all the partial runs $\{\langle \mathsf{T}_{\mathsf{pr}\,i}, \mathsf{prun}_i \rangle\}_i^n$ are 1-labeled as well.*

Proof. Item (i) We show that the three properties in Definition 9 of partial run holds.

1. Property of the root.
   The root of $\langle \mathsf{T}_{\mathsf{pr}\,i}, \mathsf{prun}_i \rangle$ is labeled by $\mathsf{prun}_i(\varepsilon) = (\varepsilon, q_i, h_i, l_i)$, since $\mathsf{prun}(y_i) = (x_i, q_i, h_i, l_i)$.
2. Property of 0-labeled nodes.
   For all $y \in \mathsf{T}_{\mathsf{pr}\,i}$, there exists $z \in \mathsf{T}_{\mathsf{pr}}$ such that $z = y_i \cdot y$, so we have that $\mathsf{prun}_i(y) = (x, q, g, 0)$, since $\mathsf{prun}(z) = (x_i \cdot x, q, g, 0)$. Moreover, since $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is a partial run, z has no successor in it, so it holds that $\mathsf{succ}_{\mathsf{T}_{\mathsf{pr}\,i}}(y) = \emptyset$.
3. Property of 1-labeled nodes.
   For all $y \in \mathsf{T}_{\mathsf{pr}\,i}$, there exists $z \in \mathsf{T}_{\mathsf{pr}}$ such that $z = y_i \cdot y$, so we have that $\mathsf{prun}_i(y) = (x, q, g, 1)$, since $\mathsf{prun}'(z) = (x_i \cdot x, q, g, 1)$. Moreover, $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is a partial run, so there exists a set $S \subseteq D_b^{\varepsilon} \times Q$, where $S \models \delta(q, g, \mathsf{inp}(x_i \cdot x))$, and a set $E \in \mathsf{exec}(S,$

$\mathsf{dir}_{\mathsf{T}}(x_i \cdot x))$ such that for all configurations $(d, q', g') \in E$ there is a node $z' = y_i \cdot y' \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(z)$ such that $\mathsf{prun}(z') = (x_i \cdot x \cdot d, q', g', l)$.
Now, since $\mathsf{inp}_i(x) = \mathsf{inp}(x_i \cdot x)$ and $\mathsf{dir}_{\mathsf{T}_i}(x) = \mathsf{dir}_{\mathsf{T}}(x_i \cdot x)$ we have that there exists a set $S \subseteq D_b^{\varepsilon} \times Q$, where $S \models \delta(q, g, \mathsf{inp}_i(x))$, and a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathsf{T}_i}(x))$ such that for all configurations $(d, q', g') \in E$ there is a node $y' \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}\,i}}(y)$ such that $\mathsf{prun}_i(y') = (x \cdot d, q', g', l)$.

Item (ii) Let us consider an infinite path $\pi \preccurlyeq \mathsf{T}_{\mathsf{pr}\,i}$. Then there exists a path $\pi' \preccurlyeq \mathsf{T}_{\mathsf{pr}}$ such that $\pi = \pi'_{\geq |y_i|}$. Since $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ is accepting and $\mathsf{inf}(\mathsf{prun}_{i|\pi}) = \mathsf{inf}(\mathsf{prun}_{|\pi'})$, we have that $\mathsf{inf}(\mathsf{prun}_{i|\pi}) \cap \mathsf{T} \times F \neq \emptyset$.

Item (iii) Finally, let us consider a node $y \in \mathsf{T}_{\mathsf{pr}\,i}$. Then there is a node $z \in \mathsf{T}_{\mathsf{pr}}$ such that $z = y_i \cdot y$, so it holds that $\mathsf{prun}'(y) = (x, q, g, 1)$, since $\mathsf{prun}_i(z) = (x_i \cdot x, q, g, 1)$, i.e., y is also 1-labeled.

Lemma 11. *Let φ be a GCTL state formula and $\mathcal{K} = \langle \mathrm{AP}, \mathrm{W}, \mathrm{R}, \mathrm{L} \rangle$ be a Kripke structure. Then, for all worlds $w \in \mathrm{W}$ and subformula $\varphi' \in \mathsf{cl}(\varphi)$ it holds that $\mathcal{K}, w \models \varphi'$ iff the unwinding $\mathcal{U}_w^{\mathcal{K}} = \langle \mathrm{AP}', \mathrm{W}', \mathrm{R}', \mathrm{L}' \rangle$ of $\mathcal{K}$ starting from w is accepted by the automaton $\mathcal{A}'_{\varphi'} = \langle \mathsf{ecl}(\varphi), 2^{\mathrm{AP}}, \mathsf{deg}(\varphi), \delta, \varphi', 0, F \rangle$. Moreover, if $\varphi' = \mathsf{E}^{\geq g}(\varphi_1 \mathsf{U} \varphi_2)$ (resp., $\mathsf{A}^{<g}(\varphi_1 \mathsf{U} \varphi_2)$, $\mathsf{E}^{\geq g}(\varphi_1 \mathsf{R} \varphi_2)$, or $\mathsf{A}^{<g}(\varphi_1 \mathsf{R} \varphi_2)$), the same unwinding is accepted by the automaton $\mathcal{A}''_{\varphi'} = \langle \mathsf{ecl}(\varphi), 2^{\mathrm{AP}}, \mathsf{deg}(\varphi), \delta, \gamma, g, F \rangle$, with $\gamma = \langle \varphi_1 \mathsf{U} \varphi_2 \rangle \in \mathsf{ecl}(\varphi)$ (resp., $[\varphi_1 \mathsf{U} \varphi_2]$, $\langle \varphi_1 \mathsf{R} \varphi_2 \rangle$, or $[\varphi_1 \mathsf{R} \varphi_2]$).*

Proof. We will show the thesis by induction on the structure of the formula φ. Note that, $\mathcal{A}'_{\varphi'}$ (resp., $\mathcal{A}''_{\varphi'}$) accepts the unwinding $\mathcal{U}_w^{\mathcal{K}}$ iff it has a run on it and so a 1-labeled partial run on it.
Base case: Atomic propositions. $\varphi' = p$ (resp., $\varphi' = \neg p$), with $p \in \mathrm{AP} = \mathrm{AP}'$.

1. If $\mathcal{K}, w \models \varphi'$ then the run of $\mathcal{A}'_{\varphi'}$ consisting of the only root is accepting, indeed we have $\delta(\varphi', 0, \mathrm{L}'(\varepsilon)) = \mathfrak{t}$ since $\delta(p, 0, \mathrm{L}'(\varepsilon)) = (p \in \mathrm{L}'(\varepsilon))$ and $p \in \mathrm{L}'(\varepsilon) = \mathrm{L}(w)$ (resp., $\delta(\neg p, 0, \mathrm{L}'(\varepsilon)) = (p \notin \mathrm{L}'(\varepsilon))$ and $p \notin \mathrm{L}'(\varepsilon) = \mathrm{L}(w)$), thus we can choose an empty set S satisfying the delta, which implies that the corresponding run will not have successors of the root and then any infinite path, so it will be accepting for definition.
2. Let us suppose that there exists an accepting run for $\mathcal{A}'_{\varphi'}$ on the unwinding tree in input. Since $\delta(p, 0, \mathrm{L}'(\varepsilon)) = (p \in \mathrm{L}'(\varepsilon))$ (resp., $\delta(\neg p, 0, \mathrm{L}'(\varepsilon)) = (p \notin \mathrm{L}'(\varepsilon))$) the only way for the tree to be accepting is that $\delta(\varphi', 0, \mathrm{L}'(\varepsilon)) = \mathfrak{t}$, thus p must be (resp., must not be) in $\mathrm{L}'(\varepsilon) = \mathrm{L}(w)$ and then $\mathcal{K}, w \models \varphi'$.

Inductive case: And (resp., Or). $\varphi' = \varphi_1 \wedge \varphi_2$ (resp., $\varphi' = \varphi_1 \vee \varphi_2$).

1. If $\mathcal{K}, w \models \varphi'$ it holds that $\mathcal{K}, w \models \varphi_1$ and $\mathcal{K}, w \models \varphi_2$ (resp., $\mathcal{K}, w \models \varphi_1$ or $\mathcal{K}, w \models \varphi_2$). For inductive hypothesis we have that both $\mathcal{A}'_{\varphi_1}$ and $\mathcal{A}'_{\varphi_2}$ have (resp., at least one between $\mathcal{A}'_{\varphi_1}$ and $\mathcal{A}'_{\varphi_2}$ has) an accepting 1-labeled partial run on the unwinding tree $\mathcal{U}_w^{\mathcal{K}}$ in input. Let $\langle \mathsf{T}_{\mathsf{pr}\,1}, \mathsf{prun}_1 \rangle$ and $\langle \mathsf{T}_{\mathsf{pr}\,2}, \mathsf{prun}_2 \rangle$ be these two partial runs (resp., let $\langle \mathsf{T}_{\mathsf{pr}\,i}, \mathsf{prun}_i \rangle$ be this partial run).
Since $\delta(\varphi_1 \wedge \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \wedge (\varepsilon, \varphi_2)$ (resp., $\delta(\varphi_1 \vee \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \vee (\varepsilon, \varphi_2)$), the transition function is satisfied by the set $S = \{(\varepsilon, \varphi_1), (\varepsilon, \varphi_2)\}$ (resp., $\{(\varepsilon, \varphi_i)\}$),

so we construct the following accepting partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$: $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon, 0, 1\}$, $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$, $\mathsf{prun}(0) = (\varepsilon, \varphi_1, 0, 0)$, $\mathsf{prun}(1) = (\varepsilon, \varphi_2, 0, 0)$ (resp., $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon, 0\}$, $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$, $\mathsf{prun}(0) = (\varepsilon, \varphi_i, 0, 0)$).
Now, extending $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ with $\langle \mathsf{T}_{\mathsf{pr}1}, \mathsf{prun}_1 \rangle$ and $\langle \mathsf{T}_{\mathsf{pr}2}, \mathsf{prun}_2 \rangle$ on 0 and 1 (resp., extending $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ with $\langle \mathsf{T}_{\mathsf{pr}i}, \mathsf{prun}_i \rangle$ on 0), by Lemma 5 we obtain an accepting 1-labeled partial run of $\mathcal{A}'_{\varphi'}$ on the unwinding.

2. Let us suppose that there exists an accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on the unwinding tree $\mathcal{U}^{\mathcal{K}}_{w}$ in input. Since $\delta(\varphi_1 \wedge \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \wedge (\varepsilon, \varphi_2)$ (resp., $\delta(\varphi_1 \vee \varphi_2, 0, \sigma) = (\varepsilon, \varphi_1) \vee (\varepsilon, \varphi_2)$), the transition function is satisfied by the set $S = \{(\varepsilon, \varphi_1), (\varepsilon, \varphi_2)\}$ (resp., $\{(\varepsilon, \varphi_i)\}$), so the root of the partial run must have two successors 0 and 1 with labels $(\varepsilon, \varphi_1, 0, 1)$ and $(\varepsilon, \varphi_2, 0, 1)$ (resp., at least the successor 0 with label $(\varepsilon, \varphi_i, 0, 1)$).
Now, consider the two trees $\langle \mathsf{T}_{\mathsf{pr}1}, \mathsf{prun}_1 \rangle$ and $\langle \mathsf{T}_{\mathsf{pr}2}, \mathsf{prun}_2 \rangle$ (resp., the tree $\langle \mathsf{T}_{\mathsf{pr}i}, \mathsf{prun}_i \rangle$) extracted from $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ on the nodes 0 and 1 (resp., on the node 0). By Lemma 6, we obtain that these two trees are (resp., this tree is an) accepting 1-labeled partial runs (resp., run) of the automata $\mathcal{A}'_{\varphi_1}$ and $\mathcal{A}'_{\varphi_2}$ (resp., of the automaton $\mathcal{A}'_{\varphi_i}$) on the same tree in input, so by inductive hypothesis it holds that $\mathcal{K}, w \models \varphi_1$ and $\mathcal{K}, w \models \varphi_2$ (resp., $\mathcal{K}, w \models \varphi_1$ or $\mathcal{K}, w \models \varphi_2$) and then $\mathcal{K}, w \models \varphi'$.

Inductive case: Exists Effective Next. $\varphi' = \mathsf{E}^{\geq g} \mathsf{X}\, \varphi''$.

1. If $\mathcal{K}, w \models \mathsf{E}^{\geq g} \mathsf{X}\, \varphi''$ it holds that $\mathcal{U}^{\mathcal{K}}_{w}, \varepsilon \models \mathsf{E}^{\geq g} \mathsf{X}\, \varphi''$. Let $\mathrm{X} = \{x \in \mathsf{succ}_{\mathcal{U}^{\mathcal{K}}_{w}}(\varepsilon) \mid \mathcal{U}^{\mathcal{K}}_{w}, x \models \varphi''\}$. By Lemma 8, $|\mathrm{X}| = |\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{U}^{\mathcal{K}}_{w}, \varepsilon, \mathsf{X}\, \varphi''))| \geq g$, so it is possible to choose a set $\mathrm{X}' = \{x_1, \ldots, x_g\} \subseteq \mathbb{N}$ of g nodes in X. By inductive hypothesis, we have that $\mathcal{A}'_{\varphi''}$ has a 1-labeled accepting partial run $\langle \mathsf{T}_{\mathsf{pr}i}, \mathsf{prun}_i \rangle$ on $x_i \lhd \mathcal{U}^{\mathcal{K}}_{w}$, for each index $i \in \mathbb{N}_{(g)+}$.
Since $\delta(\mathsf{E}^{\geq g} \mathsf{X}\, \varphi'', 0, \sigma) = (\langle g \rangle, \langle \varphi'' \rangle)$, the transition function is satisfied by the set $S = \{(\langle g \rangle, \langle \varphi'' \rangle)\}$. Now, there is a sequence of numbers $\{h_i\}_i^g \in \mathsf{CP}(g)$ with $h_1 = g$ and $h_2 = \ldots = h_g = 0$, so there is a sequence of sets $\{M_i\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir}_{\mathcal{U}^{\mathcal{K}}_{w}}(\varepsilon), \langle g \rangle)$ such that $M_1 = \mathrm{X}'$ and $M_2 = \ldots = M_{g+1} = \emptyset$. At this point, it is evident that there exists a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}^{\mathcal{K}}_{w}}(\varepsilon))$ such that $E = \{(d, \langle \varphi'' \rangle, 1) \mid d \in \mathrm{X}'\}$. Moreover $\delta(\langle \varphi'' \rangle, 1, \sigma) = (\varepsilon, \varphi'')$, so we can construct the following accepting partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on $\mathcal{U}^{\mathcal{K}}_{w}$: $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon\} \cup \mathbb{N}_{(g-1)} \cup \{i \cdot 0 \mid i \in \mathbb{N}_{(g-1)}\}$, $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$, $\mathsf{prun}(i) = (x_{i+1}, \langle \varphi'' \rangle, 1, 1)$, and $\mathsf{prun}(i \cdot 0) = (x_{i+1}, \varphi'', 0, 0)$, for $i \in \mathbb{N}_{(g-1)}$.
Now, extending $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ with $\{\langle \mathsf{T}_{\mathsf{pr}i}, \mathsf{prun}_i \rangle\}_i^g$ on $\{(i-1) \cdot 0\}_i^g$, by Lemma 5 we obtain an accepting 1-labeled partial run of $\mathcal{A}'_{\varphi'}$ on the unwinding.
2. Let us suppose that there exists an accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on the unwinding tree $\mathcal{U}^{\mathcal{K}}_{w}$ in input, with $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$.
Since $\delta(\mathsf{E}^{\geq g} \mathsf{X}\, \varphi'', 0, \sigma) = (\langle g \rangle, \langle \varphi'' \rangle)$, the transition function is satisfied by the set $S = \{(\langle g \rangle, \langle \varphi'' \rangle)\}$, so there exists a sequence $\{M_i\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir}_{\mathcal{U}^{\mathcal{K}}_{w}}(\varepsilon), \langle g \rangle)$ such that for all indexes $i \in \mathbb{N}_{(g)+}$ and directions $d \in M_i \setminus M_{i+1}$ there is a node $y \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(\varepsilon)$ such that $\mathsf{prun}(y) = (d, \langle \varphi'' \rangle, i, 1)$. Now, for all $i > 1$, $\delta(\langle \varphi'' \rangle, i, \sigma) = \mathfrak{f}$, so we have that $|M_1| = g$ and $M_2 = \ldots = M_{g+1} = \emptyset$. Moreover, $\delta(\langle \varphi'' \rangle, 1, \sigma) = (\varepsilon, \varphi'')$, so each node $y_i \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(\varepsilon)$, with $\mathsf{prun}(y_i) = (x_i, \langle \varphi'' \rangle, 1, 1)$ has a successor $y_i \cdot j$, with $j \in \mathbb{N}$, labeled by $\mathsf{prun}(y \cdot j) = (x_i, \varphi'', 0, 1)$.

Now, consider the trees $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ extracted from $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ on the nodes $y_i \cdot j$. By Lemma 6, we obtain that these trees are accepting 1-labeled partial runs of the automata $\mathcal{A}'_{\varphi''}$ on the trees $x_i \triangleleft \mathcal{U}_w^{\mathcal{K}}$, so by inductive hypothesis it holds that $\mathcal{U}_w^{\mathcal{K}}, x_i \models \varphi''$. Let $\mathrm{X} = \{x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \mid \mathcal{U}_w^{\mathcal{K}}, x \models \varphi''\}$. By Lemma 8, $|\mathsf{minstructs}(\mathfrak{P}_A(\mathcal{U}_w^{\mathcal{K}}, \varepsilon, \mathsf{X}\,\varphi''))| = |\mathrm{X}| \geq |M_1| = g$, so $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{E}^{\geq g}\mathsf{X}\,\varphi''$ and then $\mathcal{K}, w \models \mathsf{E}^{\geq g}\mathsf{X}\,\varphi''$.

Inductive case: For all Hypothetical Next. $\varphi' = \mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi''$.

1. If $\mathcal{K}, w \models \mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi''$ it holds that $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi''$. Let $\mathrm{X} = \{x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \mid \mathcal{U}_w^{\mathcal{K}}, x \models \neg\varphi''\} = \{x'_1, \ldots, x'_{|\mathrm{X}|}\}$. By Lemma 8, $|\mathrm{X}| = |\mathsf{minstructs}(\mathsf{paths}(\mathcal{U}_w^{\mathcal{K}}, \varepsilon) \setminus \mathfrak{P}_E(\mathcal{U}_w^{\mathcal{K}}, \varepsilon, \tilde{\mathsf{X}}\,\varphi''))| < g$, so it is possible to choose the set $\mathrm{X}' = \{x_1, x_2, \ldots\} = \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \setminus \mathrm{X} \subseteq \mathbb{N}$. By inductive hypothesis, we have that $\mathcal{A}'_{\varphi''}$ has a 1-labeled accepting partial run $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ on $x_i \triangleleft \mathcal{U}_w^{\mathcal{K}}$, for each index $i \in \mathbb{N}_{(|\mathrm{X}'|)+}$.
   Since $\delta(\mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi'', 0, \sigma) = ([g], [\varphi''])$, the transition function is satisfied by the set $S = \{([g], [\varphi''])\}$. Now, there is a sequence of numbers $\{h_i\}_i^g \in \mathsf{CP}(g)$ with $h_1 = g$ and $h_2 = \ldots = h_g = 0$, so there is a sequence of sets $\{M_i\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon), [g])$ such that $M_1 = \mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon)$, $M_2 = \mathrm{X}$, and $M_3 = \ldots = M_{g+1} = \emptyset$. At this point, it is evident that there exists a set $E \in \mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon))$ such that $E = \{(d, [\varphi''], 1) \mid d \in \mathrm{X}'\} \cup \{(d, [\varphi''], 2) \mid d \in \mathrm{X}\}$. Moreover $\delta([\varphi''], 1, \sigma) = (\varepsilon, \varphi'')$ and $\delta([\varphi''], 2, \sigma) = \mathfrak{t}$, so we can construct the following accepting partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on $\mathcal{U}_w^{\mathcal{K}}$: $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon\} \cup \mathbb{N}_{(|\mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon)|-1)} \cup \{(i + |\mathrm{X}|) \cdot 0 \mid i \in \mathbb{N}_{(|\mathrm{X}'|-1)}\}$, $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$, $\mathsf{prun}(i) = (x'_{i+1}, [\varphi''], 2, 1)$, $\mathsf{prun}(j + |\mathrm{X}|) = (x_{j+1}, [\varphi''], 1, 1)$, and $\mathsf{prun}((j + |\mathrm{X}|) \cdot 0) = (x_{j+1}, \varphi'', 0, 0)$, for $i \in \mathbb{N}_{(|\mathrm{X}|-1)}$ and $j \in \mathbb{N}_{(|\mathrm{X}'|-1)}$.
   Now, extending $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ with $\{\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle\}_i^{|\mathrm{X}'|}$ on $\{(i + |\mathrm{X}| - 1) \cdot 0\}_i^{|\mathrm{X}'|}$, by Lemma 5 we obtain an accepting 1-labeled partial run of $\mathcal{A}'_{\varphi'}$ on the unwinding.
2. Let us suppose that there exists an accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on the unwinding tree $\mathcal{U}_w^{\mathcal{K}}$ in input, with $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$.
   Since $\delta(\mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi'', 0, \sigma) = ([g], [\varphi''])$, the transition function is satisfied by the set $S = \{([g], [\varphi''])\}$, so there exists a sequence $\{M_i\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon), [g])$ such that for all indexes $i \in \mathbb{N}_{(g)+}$ and directions $d \in M_i \setminus M_{i+1}$ there is a node $y \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(\varepsilon)$ such that $\mathsf{prun}(y) = (d, [\varphi''], i, 1)$. Note that $|M_2| < g$. Moreover, $\delta([\varphi''], 1, \sigma) = (\varepsilon, \varphi'')$, so each node $y_i \in \mathsf{succ}_{\mathsf{T}_{\mathsf{pr}}}(\varepsilon)$, with $\mathsf{prun}(y_i) = (x_i, [\varphi''], 1, 1)$ has a successor $y_i \cdot j$, with $j \in \mathbb{N}$, labeled by $\mathsf{prun}(y \cdot j) = (x_i, \varphi'', 0, 1)$.
   Now, consider the trees $\langle \mathsf{T}_{\mathsf{pr}_i}, \mathsf{prun}_i \rangle$ extracted from $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ on the nodes $y_i \cdot j$. By Lemma 6, we obtain that these trees are accepting 1-labeled partial runs of the automata $\mathcal{A}'_{\varphi''}$ on the trees $x_i \triangleleft \mathcal{U}_w^{\mathcal{K}}$, so by inductive hypothesis it holds that $\mathcal{U}_w^{\mathcal{K}}, x_i \models \varphi''$. Let $\mathrm{X} = \{x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \mid \mathcal{U}_w^{\mathcal{K}}, x \models \neg\varphi''\}$. By Lemma 8, $|\mathsf{minstructs}(\mathsf{paths}(\mathcal{K}, w) \setminus \mathfrak{P}_E(\mathcal{U}_w^{\mathcal{K}}, \varepsilon, \tilde{\mathsf{X}}\,\varphi''))| = |\mathrm{X}| \leq |M_2| < g$, so $\mathcal{U}_w^{\mathcal{K}}, \varepsilon \models \mathsf{A}^{<g}\tilde{\mathsf{X}}\,\varphi''$ and then $\mathcal{K}, w \models \mathsf{A}^{<g}\mathsf{X}\,\varphi''$.

Inductive case: Does not exist a successor. $\varphi' = \mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f}$.

1. If $\mathcal{K}, w \models \mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f}$, by Lemma 7, it holds that $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) = \emptyset$. Then, we can construct the following accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on $\mathcal{U}_w^{\mathcal{K}}$: $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon\}$

and $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$. This partial run is also a valid run. Indeed, $\delta(\mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f}, 0, \sigma) = ([1], \mathfrak{f})$, so we can choose the set $S = \{([1], \mathfrak{f})\}$ and then, accordingly to $\mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon))$, for all $x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon)$ it holds that ε has a successor with label $(x, \mathfrak{f}, 1, 1)$, but $\delta(\mathfrak{f}, 1, \sigma) = \mathfrak{f}$, so the construction is correct since $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) = \emptyset$.

2. Let us suppose that there exists an accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on the unwinding tree $\mathcal{U}_w^{\mathcal{K}}$ in input, with $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$.
   Since $\delta(\mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f}, 0, \sigma) = ([1], \mathfrak{f})$, the transition function is satisfied by the set $S = \{([1], \mathfrak{f})\}$, so accordingly to $\mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon))$, for all $x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon)$ it holds that ε has a successor with label $(x, \mathfrak{f}, 1, 1)$, but $\delta(\mathfrak{f}, 1, \sigma) = \mathfrak{f}$, so it must hold that $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) = \emptyset$ then, by Lemma 7, $\mathcal{K}, w \models \mathsf{E}\tilde{\mathsf{X}}\,\mathfrak{f}$.

Inductive case: There exists a successor. $\varphi' = \mathsf{AX}\,\mathfrak{t}$.

1. If $\mathcal{K}, w \models \mathsf{AX}\,\mathfrak{t}$, by Lemma 7, it holds that $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \neq \emptyset$. Then, we can construct the following accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on $\mathcal{U}_w^{\mathcal{K}}$: $\mathsf{T}_{\mathsf{pr}} = \{\varepsilon, 0\}$, $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$, and $\mathsf{prun}(0) = (x, \mathfrak{t}, 1, 1)$, with $x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon)$. This partial run is also a valid run.
   Indeed, $\delta(\mathsf{AX}\,\mathfrak{t}, 0, \sigma) = (\langle 1 \rangle, \mathfrak{t})$, so we can choose the set $S = \{(\langle 1 \rangle, \mathfrak{t})\}$ and then, accordingly to $\mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon))$, there exists $x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon)$ such that ε has a successor 0 with label $(x, \mathfrak{t}, 1, 1)$. Moreover, since $\delta(\mathfrak{t}, 1, \sigma) = \mathfrak{t}$, we can choose the set $S = \emptyset$ and thus 0 does not need to have any successor.
2. Let us suppose that there exists an accepting 1-labeled partial run $\langle \mathsf{T}_{\mathsf{pr}}, \mathsf{prun} \rangle$ for $\mathcal{A}'_{\varphi'}$ on the unwinding tree $\mathcal{U}_w^{\mathcal{K}}$ in input, with $\mathsf{prun}(\varepsilon) = (\varepsilon, \varphi', 0, 1)$.
   Since $\delta(\mathsf{AX}\,\mathfrak{t}, 0, \sigma) = (\langle 1 \rangle, \mathfrak{t})$, the transition function is satisfied by the set $S = \{(\langle 1 \rangle, \mathfrak{t})\}$, so accordingly to $\mathsf{exec}(S, \mathsf{dir}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon))$, there exists $x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon)$ such that ε has a successor 0 with label $(x, \mathfrak{t}, 1, 1)$, with $x \in \mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon)$, so it must hold that $\mathsf{succ}_{\mathcal{U}_w^{\mathcal{K}}}(\varepsilon) \neq \emptyset$ then, by Lemma 7, $\mathcal{K}, w \models \mathsf{AX}\,\mathfrak{t}$.

E Construction and correctness proof of Theorem 4

Definition 12. *A 2-player parity game is a structure* $((V, A), \{V_1, V_2\}, v_0, \{W_1, W_2, \ldots, W_k\})$ *where*

1. *(V, A) is an oriented graph, with V the set of nodes and $A \subseteq V \times V$ the set of edges. The nodes are also called states of the game.*
2. *$\{V_1, V_2\}$ is a partition of V in two sets.*
3. *$v_0 \in V$ is a node called the starting state.*
4. *$\{W_1, W_2, \ldots, W_k\}$ is a partion of V in k sets, this partition is called the winning condition.*

Definition 13. *In a 2-player parity game* $((V, A), \{V_1, V_2\}, v_0, \{W_1, W_2, \ldots, W_k\})$ *a match is a finite or infinite sequence of states $s = \{s_i\}_i$ such that*

1. *$s_0 = v_0$*

2. *For all i* $(s_i, s_{i+1}) \in A$

So a match is a path of (V,A) *starting at* v_0.

A match can be finite only if it's last state s_k *does not have a successor, i.e., if* $|\{s \in V \mid (s_k, s) \in A\}| = 0$.

A match is always constructed iteratively: the match start with the only state $s_0 = v_0$. *At every step i the last node added is* s_i*, if* $s_i \in V_1$ *the player* 1 *chooses the next state* $s_{i+1} \in \{s \in V \mid (s_i, s) \in A\}$*, if* $s_i \in V_2$ *the player* 2 *chooses the next state. If a player can make a move, he must make a move, so the match may be finite only if a player cannot make any more moves.*

Definition 14. *In a 2-player parity game* $((V,A), \{V_1,V_2\}, v_0, \{W_1,W_2,\dots,W_k\})$ *a finite match* $s = s_0,\dots,s_k$ *is winning for player* 1 *iff its last state* s_k *is in* V_2.

Definition 15. *In a 2-player parity game* $((V,A), \{V_1,V_2\}, v_0, \{W_1,W_2,\dots,W_k\})$ *a infinite match* $s = s_0,\dots,s_k,\dots$ *is winning for player* 1 *iff the least index i, such that s visits infinitely often states in* W_i*, is odd, i.e., the least i such that* $|\{j \in N \mid s_j \in W_i\}| = +\infty$ *is odd*

Definition 16. *In a 2-player parity game a strategy for player i is a function* $st_i : \cup_{j=0}^{+\infty}(V^j \times V_i) \rightarrow V$*, that for any starting match* $s_0,\dots,s_k$*, with* $s_k \in V_i$*, tells to player i what his next move is, by giving him the next state of the match.*

Definition 17. *In a 2-player parity game a match s is saied to follow the strategy* st_1 *and* st_2 *if for all player i and for all* $k \in N$ *such that* $s_k \in V_i$ *we have* $st_i(s_0,\dots,s_k) = s_{k+1}$. *It is obvious that for any pair of strategy* st_1, st_2 *there exists only one match following them.*

Definition 18. *In a 2-player parity game a strategy* st_i *for player i is winning for player i iff for all strategy* st_j *for the other player j, the match following the strategies* st_i, st_j *is winning for player* 1.

Definition 19. *In a 2-player parity game a strategy* st_i *for player i is memoryless, if there exists a function* $st'_i : V_i \rightarrow V$ *such that for all* $s_0,\dots,s_k \in \cup_{j=0}^{+\infty}(V^j \times V_i)$ $st_i(s_0,\dots,s_k) = st'_i(s_k)$. *So the strategy is memoryless if it depends only on the last state of the game, in this situation the function* st'_i *is used to represet memoryless strategies.*

Theorem 5. *Jutla's Theorem*

In a 2-player parity game there exists a winning strategy st_i *for player i iff there exists a winning memoryless strategy* st'_i *for player i.*

Lemma 12. *For a PABT* $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F\rangle$*, there exists an accepting run on the input* $\langle \mathsf{T}, \mathsf{inp}\rangle$ *iff there exists a memoryless anccepting run on* $\langle \mathsf{T}, \mathsf{inp}\rangle$.

Proof. Let's construct the following 2-player parity game $((V,A), \{V_1,V_2\}, v_0, \{W_1,W_2\})$, with:

1. $V_1 = \{(x,q,b) \in T \times Q \times N_b\} \cup \{(x,s) \in T \times (D_b \times Q)\}$,

2. $V_2 = \{(x,S) \in T \times B + (D_b \times Q)\} \cup \{(x,q',\{M_i\}_i) \in T \times Q \times D_b \times \cup_{b \in D_b}(\mathsf{spart}(\mathsf{dir}_\mathsf{T}(x), b))\}$,
3. $V = V_1 \cup V_2$,
4. $F' = \{(x,q,b) \in T \times F \times N_b)\}$ is the set of configuration with accepting states,
5. v_0=(ε, q_0, g_0),
6. $W_1 = F'$,
7. $W_2 = V - F'$,
8. A is the union of the following sets:
   (a) $\{((x,q,b),(x,S)) \in V \times V \mid S \models \delta(q,b,\mathsf{inp}(x))\}$
   (b) $\{((x,S),(x,s)) \in V \times V \mid s \in S\}$
   (c) $\{((x,S),(x,q',0)) \in V \times V \mid (\varepsilon,q') \in S\}$
   (d) $\{((x,(q',<g>)),(x,q',\{M_i\}_i)) \in V \times V \mid \{M_i\}_i \in \mathsf{spart}(\mathsf{dir}_\mathsf{T}(x),<g>)\}$
   (e) $\{((x,(q',[g])),(x,q',\{M_i\}_i)) \in V \times V \mid \{M_i\}_i \in \mathsf{spart}(\mathsf{dir}_\mathsf{T}(x),[g])\}$
   (f) $\{((x,q',\{M_i\}_i),(x',q',i)) \in V \times V \mid x' \in M_i - M_{i+1}\}$

The proof is composed by three steps.

Lemma 13. *First we show that a sequence $r = \{r_i\}_i$ of configurations of $(T \times Q \times D_b)^* \cup (T \times Q \times D_b)^w$ is a sequence of labels of an accepting branch of a run for $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp}\rangle$ iff there exists a winning match $s = \{s_j\}_j \in V * \cup V^w$ for player 1 such that the subsequence of s made only by the nodes of $\{(x,q,b) \in T \times Q \times N_b\}$ is r.*

Proof. *(i)* First let $r = \{r_i\}_i$ be a sequence of labels of an accepting branch π of a run for $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp}\rangle$, such that $\mathsf{inp}(\pi(i)) = r_i$. Then we construct the match $s = \{s_i\}_i$. The match start at the state $s_0 = (\varepsilon, q_0, g_0)$. This state is also the root of any run of $\mathcal{A}$, so it is the first node of any accepting branch and it is equal to r_0.

Now as induction hypothesis we suppose that the match constructed so far is given by
$s_0, s_1, \ldots, s_k$, with $s_0 = r_0$, $r_j = s_{i(j)}$ such that $i(j) = min\{i > i(j-1) \mid s_i \in \{(x,q,b) \in T \times Q \times N_b\}\}$, and $s_k \in \{(x,q,b) \in T \times Q \times N_b\}\}$.

Then we show that we can keep constructing the match, while preserving the induction properties. Precisely, we show that one of the following statement holds.

1. If the node $\pi(i^{-1}(k))$ with label $r_{i^{-1}(k)}$ does not have a successor in π then we can add only one more state to the match s, and we show that the match is winning for player 1.
2. If the node $\pi(i^{-1}(k))$ has a successor in π then we can add more states to the match s and, after this operation, the obtained extended match satisfy the induction hypothesis.

Let's show the above statements. In the following proofs we assume that $\mathsf{inp}(\pi(i^{-1}(k))) = (x,q,b)$, and $\mathsf{inp}(\pi(i^{-1}(k)+1)) = (x',q',b')$, with $x' = xd$.

1. If $\pi(i^{-1}(k))$ does not have a successor in π, then $\pi(i^{-1}(k))$ has no successor at all, else π would not be a valid branch. (By definition a branch is allowed to be finite only if its last node does not have a successor). So, by the properties of a run for $\mathcal{A}$ we have $\delta(q,b,\mathsf{inp}(x)) = true$. Hence, during the construction of the run,

$S = \emptyset$ was the choosen sodisfacibility set, and $\emptyset \in \mathsf{exec}(S, \mathsf{dir}_\mathsf{T}(x))$ was used for the construction of the successors of $\pi(i^{-1}(k))$. So player 1 can make a move, by adding to the match the state $(x,S) \in V_2$. Now player 2 cannot make any move since S is empty, so player 1 wins.

2. If $\pi(i^{-1}(k))$ have a successor in π, then there exists a sodisfacibility set $S \models \delta(q, b, \mathsf{inp}(x))$ so player 1 can make a move, by adding to the match the state $(x,S) \in V_2$.
   There are two possible situations.
   (a) If $x = x'$, then necessarily $b' = \varepsilon$ since the run properties hold. So, $(\varepsilon, q') \in S$ and player 2 can make a move by adding to the match S the state $(x', q', b') \in V_1$. Now the match S was extended with two more states, and it still satisfy the induction hypothesis.
   (b) If x' is a successor of x in $\langle \mathsf{T}, \mathsf{inp} \rangle$, then $b' \neq \varepsilon$ and there exists $b'' \mid (b'', q') \in S$ such that there exists a sequence $\{M_i\}_i \in \mathsf{spart}(\mathsf{dir}_\mathsf{T}(x), b'')$. So, player 2 can make a move by adding to the match s the state $(x, (b'', q')) \in V_1$. Then, player 1 can add the state $(x, q', \{M_i\}_i)$ since $\{M_i\}_i \in \mathsf{spart}(\mathsf{dir}_\mathsf{T}(x), b'')$, and player 2 can add the state $(xd, q', b') \in V_1$ where b' is the index such that $d \in M_{b'} - M_{b'+1}$. Now the match S was extended with four more states, and it still satisfy the induction hypothesis.

So with a possibly infinite iteration we can build the whole match s such that r is a subsequence of s.

Now we show that the match s is winning for player 1 even if it's infinite.

Since π is accepting, there exists a state of F visited infinitely often, so, the labels in F' are visited infinitely often, i.e., $|\{i \in N \mid \mathsf{inp}(\pi(i)) = r_i \in F'\}| = +\infty$.

Since r is a subsequence of s we have that s visits infinitely often states of $F' = W_1$.

Since the least index i, such that s visits infinitely often states in W_i, is odd, player 1 wins the match s.

(ii) Let $s = \{s_i\}_i$ be a winning match for player 1, and let $r = \{r_j\}_j = \{s_{i(j)}\}_j$ be the subsequence such that $i(0) = 0$ and $i(j) = min\{k > i(j-1) \mid s_k \in \{(x,q,b) \in T \times Q \times N_b\}\}$. Then we can construct a run of $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp} \rangle$ such that there exists a branch π with $\mathsf{inp}(\pi(i)) = r_i$ for all i. For brevity we will construct only the branch π. The branch starts at the root of the run, so $\pi(0)$ has label (ε, q_0, g_0) which is the match starting state $r_0 = s_0$.

Now as induction hypothesis we suppose that the branch constructed so far is $\pi(0), \ldots, \pi(k)$ with $\mathsf{inp}(\pi(i)) = r_i$ for all i.

Then we show that we can keep constructing the branch, while preserving the induction properties. Precisely, we show that one of the following statement holds.

1. If r has length k, then the branch ends at $\pi(k)$ and it is accepting.
2. If r_{k+1} exists then we can extend the branch by adding another node $\pi(k+1)$ with label r_{k+1}.

Let's show the above statements. In the following proofs we assume that $r_k = (x,q,b)$, and $r_{k+1} = (x', q', b')$ with $x' = xd$.

1. If r ends at r_k then the match s ends at a state s_l with $l > i(k)$ such that $s_{i(k)+1}, \ldots, s_l$ are not in $T \times Q \times D_b$. Since by hypothesis s is winning for player 1, $s_l \in V_2$.
   There are two possible ends for the match s.

(a) $s_{i(k)+1}, \dots, s_l = (x,q,b), (x,S), (x,s), (x,q',\{M_i\}_i)$. Player 2 cannot make a move only if $\forall i \in N_{g+}\ M_i - M_{i+1} = \emptyset$, but this is impossible because of the properties of $\{M_i\}_i$, required in the last move of player 1. So, this end of the match is not allowed.
(b) $s_{i(k)+1}, \dots, s_l = (x,q,b), (x,S)$. Player 2 cannot make a move only if $S = \emptyset$, so it must be $\delta(q,b,\mathsf{inp}(x)) = true$. So the branch π cannot be extend in the run, $\pi(k)$ has no successors and the finite branch is accepting.

2. If r_{k+1} exists, then we have $s_{i(k)} = r_k = (x,q,b)$ and $s_{i(k)+1} = (x,S)$ such that $S \models \delta(q,b,\mathsf{inp}(x))$. So we can choose S as the sodisfacibility set required for the construction of the successors of $\pi(k)$.
Now there are two sitations.
   (a) It can be $s_{i(k)+2} = (x',q',b') = r_{k+1}$, with $x' = x$ and $b' = \varepsilon$, so, we have $(\varepsilon,q') \in S$. Then, we can add a successor to $\pi(k)$ with label (x',q',b') and we call this new node $\pi(k+1)$.
   (b) It can be $s_{i(k)+2} = (x,(b'',q'))$ with $(b'',q') \in S$ and $b'' \neq \varepsilon$. $s_{i(k)+3} = (x,q',\{M_i\}_i)$ with $\{M_i\}_i \in \mathsf{spart}(\mathsf{dir}_\mathsf{T}(x),b'')$, $s_{i(k)+4} = (xd,q',b')$, with $d \in M_{b'} - M_{b'+1}$. Then, we can add a successor to $\pi(k)$ with label (xd,q',b').

So iteratively we construct the finite or infinite branch π such that for all i $\mathsf{inp}(\pi(i)) = r_i$.

We alredy showed that if π is finite then it is accepting, so we need to show that π is accepting even if it is infinite.

If π is infinte, then r and s are infinite too. Since s is winning for player 1 the least index i, such that s visits infinitely often states in W_i, is odd and, so, it is 1.

So in the branch π the nodes have label in $F' = W_1$ infinitely often, since the state of $\mathcal{A}$ are finite, there exists a state of F visited infinitely often in π, so, the branch is accepting.

Lemma 14. *We show that player* 1 *has a winning strategy iff there exists an accepting run for* $\mathcal{A}$ *on* $\langle \mathsf{T}, \mathsf{inp} \rangle$*, moreover, player* 1 *has a memoryless winning strategy iff there exists a memoryless accepting run for* $\mathcal{A}$ *on* $\langle \mathsf{T}, \mathsf{inp} \rangle$.

Proof. (i) If there exists an accepting run $\langle \mathsf{T_r}, \mathsf{run} \rangle$ for $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp} \rangle$, then we can construct the following strategy for player 1. We define the strategy only for any starting match $s = s_0, \dots, s_k$ with $s_k \in V_1$, such that there exists a branch π of the run, and a numberb $k' \in N$ such that the sequence $\mathsf{run}(\pi(0)), \dots, \mathsf{run}(\pi(k'))$ is the subsequence of states of s in $\{(x,q,b) \in T \times Q \times N_b\}$. For any other starting match, the strategy is unimportant and may associate any fixed state.

1. If $s_k = (x,q,b)$, then $s_k = \pi(k')$ and player 1 can make a move by adding to the match a state (x,S) with S the sodisfacibility set of $\delta(q,b,\mathsf{inp}(x))$ choosen for the construction of the successors of $\pi(k')$.
2. If $s_{k-2} = (x,q,b)$, $s_{k-1} = (x,S)$, and $s_k = (x,s)$, with $s = (q',b'') \in S$, then $\pi(k') = s_{k-2}$ and player 1 can make a move by adding to the match a state $(x,q',\{M_i\}_i)$ with $\{M_i\}_i \in \cup_{g \in D_b - \{\varepsilon\}}(\mathsf{spart}(\mathsf{dir}_\mathsf{T}(x),g))$ the sequence of sets choosen for the construction of the successors of $\pi(k')$.

No we show that for every match $s = \{s_i\}$ following the above strategy st_1 and any strategy st_2 there exists a branch π such that the sequence of labels $\{\mathsf{run}\pi(i)\}_i$ is the subsequence of s of states in $\{(x,q,b) \in T \times Q \times N_b\}$.

This is because every match start at the state (ε, q_0, g_0), moreover, if for the starting match $s_0, \ldots, s_k$ with $s_k = (x,q,b)$ there exists a starting branch π such that $\mathsf{run}(\pi(0)), \ldots, \mathsf{run}(\pi(k'))$ is the subsequence of s of states in $\{(x,q,b) \in T \times Q \times N_b\}$, the two only two events may happen.

1. Player 1 moves in state (x,S) according to his strategy, player 2 moves in state a $(x,q',0)$ because $(\varepsilon,q') \in S$, so, there exists a successor of $\pi(k')$ with label $(x',q',0)$ and the match keeps following the labels of a branch.
2. Player 1 moves in state (x,S) according to his strategy, player 2 chooses a set $s = (q',b'') \in S$ and move in state (x,s), player 1 moves in state $(x,q',\{M_i\}_i)$ according to his strategy, then player 2 can only move in a state (xd,q',b') such that $d \in M_{b'} - M_{b'+1}$, so there exists a successor of $\pi(k')$ with label (xd,q',b') and the match keeps following the labels of a branch.

Since for every match s following st_1, there exists a branch π of the accepting run such that $\{\mathsf{run}\pi(i)\}_i$ is the subsequence of s in $\{(x,q,b) \in T \times Q \times N_b\}$, and since every branch is acccepting then by the first step of the proof every match following st_1 is winning for player 1.

If the run for $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp} \rangle$ is memoryless, then at every node $y \in \mathsf{T_r}$ with $\mathsf{run}y = (x,q,b)$, in order to construct the successor of y, the same satisfability set S was always choosen and for any $s \in S$ the same sequence of sets $\{M_i\}_i$ was always choosen.

So, at any state (x,q,b) of a match, the strategy st_1 makes player 1 make a move to the same state (x,S) regardless of the previuos states of the match; at any state $(x,(q,b''))$ of a match, the strategy st_1 makes player 1 make a move to a state $(x,q',\{M_i\}_i)$ regardless of the previuos states of the match.

So if $\langle \mathsf{T_r}, \mathsf{run} \rangle$ is memoryless, then st_1 is memoryless.

(ii) If there exists a winning strategy st_1 for player 1, the we can construct the following run for $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp} \rangle$.

The run starts at the root with label (ε, q_0, g_0). Then we keep constructing the run iterativily, at each i step we construct the successors of the nodes at level $i-1$ of the run.

During the iteration we use an inductive hypothesis: for every starting branch $\pi = \pi(0), \ldots, \pi(k)$ with $k \leq i-1$ there exists a match $s = s_0, \ldots, s_{k'}$ following st_1 such that $\{\mathsf{run}\pi(i)\}_i$ is the subsequence of s of states in $\{(x,q,b) \in T \times Q \times N_b\}$, and $\mathsf{run}(\pi(k)) = s_{k'}$.

Let's construct the successor of a node $\pi(k)$ with $k = i-1$ and $\mathsf{run}(\pi(k)) = (x,q,b)$. Since $st_1(s_0, \ldots, s_{k'}) = (x,S)$, we know that $S \models \delta(q,b,\mathsf{inp}(x))$ and we can choose S as the sodisfacibility set for the construction of the successors of $\pi(k)$.

1. If $S = \emptyset$ the node $\pi(k)$ has no successor.
2. For all pairs $(\varepsilon,q') \in S$, we must add a successor $\pi(k+1)$ to $\pi(k)$ with label $(x,q',0)$. In this situations there exists the starting match $s' = s_0, \ldots, s_k, (x,S), (x,q',0)$ such that $\mathsf{run}(\pi(0)), \ldots, \mathsf{run}(\pi(k+1))$ is the subsequence of s' of states in $\{(x,q,b) \in T \times Q \times N_b\}$, and $\mathsf{run}(\pi(k+1)) = (x,q',0)$

3. For all pairs $s = (b', q') \in S$ with $b' \neq \varepsilon$, we can choose the sequence of sets $\{M_i\}_i$ such that $(x, q', \{M_i\}_i) = st_1(s_0, \ldots, s_k, (x, S), (x, s))$. The for all b' and for all $d \in M_{b'} - M_{b'+1}$ we add to $\pi(k)$ a successor $\pi(k+1)$ with label (xd, q', b'). In this situations there exists the starting match $s' = s_0, \ldots, s_k, (x, S), (x, s), (x, q', \{M_i\}_i), (xs, q', b')$ such that $\mathsf{run}(\pi(0)), \ldots, \mathsf{run}(\pi(k+1))$ is the subsequence of s' of states in $\{(x, q, b) \in T \times Q \times N_b\}$, and $\mathsf{run}(\pi(k+1)) = (xd, q', b')$

So iterativily we construct the run $\langle \mathsf{T_r}, \mathsf{run} \rangle$ for $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp} \rangle$, moreover by costruction we have that for every branch π of the run there exists a match s following st_1 such that $\{\mathsf{run}(\pi(i))\}_i$ is the subsequence of s of states in $\{(x, q, b) \in T \times Q \times N_b\}$.

Since every match s following st_1 is winning for player 1, by the lemma's previous step every branch of $\langle \mathsf{T_r}, \mathsf{run} \rangle$ is accepting, so, $\langle \mathsf{T_r}, \mathsf{run} \rangle$ is accepting too.

If st_1 is memoryless at any state (x, q, b) of a match, the strategy st_1 makes player 1 move to the same state (x, S) regardless of the previuos states of the match; at any state $(x, (q, b''))$ of a match, the strategy st_1 makes player 1 move to a state $(x, q', \{M_i\}_i)$ regardless of the previuos states of the match.

So during the construction of $\langle \mathsf{T_r}, \mathsf{run} \rangle$, when we construct the successor of a node y with label $\mathsf{run}(y) = (x, q, b)$, we always choose the same satisfability set S and for all $s \in S$ we always choose the same sequence of sets $\{M_i\}_i$.

So if st_1 is memoryless, then $\langle \mathsf{T_r}, \mathsf{run} \rangle$ is memoryless.

Lemma 15. *By theorem 5 player* 1 *has a winning strategy iff he has a memoryless winning strategy. So, $\mathcal{A}$ has an accepting run on $\langle \mathsf{T}, \mathsf{inp} \rangle$ iff there exists a winning strategy for player* 1*, so, iff there exists a memoryless winning stratefy for player* 1*, and, so, iff $\mathcal{A}$ has a memoryless accepting run on $\langle \mathsf{T}, \mathsf{inp} \rangle$.*

Lemma 16. *If there exists an input tree $\langle \mathsf{T}, \mathsf{inp} \rangle$ accepted by a PABT $\mathcal{A} = \langle Q, \Sigma, b, \delta, q_0, g_0, F \rangle$, then there exists an input tree $\langle \mathsf{T}, \mathsf{inp} \rangle$ accepted by $\mathcal{A}$ such that for every node $x \in \mathsf{T}'$ the number of successors of x is at most $D = |Q| \frac{b(b-1)}{2}$, i.e., $\mathsf{T} \subseteq [D-1]^*$.*

Proof. Since $\langle \mathsf{T}, \mathsf{inp} \rangle$ is accepted by $\mathcal{A}$ by lemma 12, there exists a memoryless run $\langle \mathsf{T_r}, \mathsf{run} \rangle$ for $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp} \rangle$.

For every node $x \in \mathsf{T}$ we define the set $M(x) = \{y \in \mathsf{T_r} \mid \exists q \in Q, b \in N_b \mid \mathsf{run}(y) = (x, q, b)\}$, so, it contains all and only the nodes of $\mathsf{T_r}$ which have x as first component of their labels.

For all $y \in M(x)$, with $\mathsf{run}(y) = (x, q, b)$ we call $S(y)$ the satisfability set of the formula
$\delta(q, b, \mathsf{inp}(x))$.

Since $\langle \mathsf{T_r}, \mathsf{run} \rangle$ is memoryless, for all $(<g>, q') \in S(y)$, we choose always the same sets of directions $\{M_{i,<g>,q'}\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir_T}(x), <g>)$ in order to construct the labels of the successors of y.

Indeed, for all $d \in M_{i,<g>,q'} - M_{i+1,<g>,q'}$ there exists a successor of y labeled with (xd, q', i).

By definition of $\mathsf{spart}(\mathsf{dir_T}(x), <g>)$ there exists a sequence $\{h_i\}_i \in CP(g)$ such that for all $i \in N_g$ $|M_{i,<g>,q'}| = h_i$, $|M_{g+1,<g>,q'}| = 0$, and $M_{g,<g>,q'} \subseteq \ldots \subseteq M_{1,<g>,q'}$. So, $\sum_{i=1}^{g} |M_{i,<g>,q'} - M_{i+1,<g>,q'}| = \sum_{i=1}^{g-1} h_i - h_{i+1} + h_g \leq g$.

Hence, the set $Dc(x) = \cup_{(<g>,q')\in(D_b\times Q)}(M_{i,<g>,q'} - M_{i+1,<g>,q'})$ has cardinality $|Dc(x)| \le \sum_{(<g>,q')\in(D_b\times Q)} g = |Q|\sum_{g=1}^{b} g = |Q|\frac{b(b-1)}{2} = D$

We construct the new input tree $\langle \mathsf{T}', \mathsf{inp}' \rangle$ by removing from $\langle \mathsf{T}, \mathsf{inp} \rangle$ all the successors of nodes that don't belog to some set $Sc(x) = \{x' \in \mathsf{succ}_{\mathsf{T}}(x) \mid \exists d \in Dc(x) \mid x' = xd\}$ (unless they are the root). Precisely $\mathsf{T}' = \mathsf{T} - \{x'' \in \mathsf{T} \mid \exists x' \in \mathsf{T} - \{\varepsilon\}, z \in N* \mid (x'' = x'z) \wedge (\nexists x \in \mathsf{T} \mid x' \in Sc(x))\}$, and for all $x \in \mathsf{T}'$ $\mathsf{inp}'(x) = \mathsf{inp}(x)$.

The we construct an accepting run $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ for $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp} \rangle$ by removing from $\langle \mathsf{T_r}, \mathsf{run} \rangle$ every node y such that its label's first component is not in T'. Precisely $\mathsf{T_r}' = \mathsf{T_r} - \{y \in \mathsf{T_r} \mid \exists x \in \mathsf{T} - \mathsf{T}', q \in Q, d \in N_b \mid \mathsf{run}(y) = (x, q, d)\}$, and for all $y \in \mathsf{T_r}'$ $\mathsf{run}'(y) = \mathsf{run}(y)$.

Now we have only to show that $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ is effectively a run of $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp} \rangle$, by showing that the properties between a node in the run and its successor hold.

For every node $y \in \mathsf{T_r}' \subseteq \mathsf{T_r}$ labeled with $\mathsf{run}(y) = (x, q, b)$, by construction of $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ we have $x \in \mathsf{T}'$.

So, in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ we can use the same sodisfacibility set $S(y) \models \delta(q, b, \mathsf{inp}(x))$ choosen for the construction of the successor of y in $\langle \mathsf{T_r}, \mathsf{run} \rangle$, then, for all $s \in S(y)$ according to $exec_{\mathsf{T}}(x, S(y))$, in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$, y must have some specific successors.

Below we show that y has the successors required for all $s \in S(y)$.

1. If $s = (\varepsilon, q')$, then in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ there exists a successor y' of y with label $\mathsf{run}(y') = (x, q', 0)$, since $x \in \mathsf{T}'$, then, $y' \in \mathsf{T_r}'$.
2. If $s = (<g>, q')$ then for all $x' \in M_{i,<g>,q'} - M_{i+1,<g>,q'}$ there exists a successor $y' \in \mathsf{T_r}$ of y such that $\mathsf{run}(y') = (x', q', i)$. We easily see that every direction $d \in M_{i,<g>,q'} - M_{i+1,<g>,q'}$ is such that $xd \in \mathsf{T}'$: this is because $d \in Dc(x)$ and x is not a descendant of any node x'' such that do not exists x''' such that $x'' = x'''d'$ and $d' \notin Dc(x''')$. Hence, every node y' is a node of $\mathsf{T_r}'$.
3. If $s = ([g], q')$, then, there exists a sequence $\{M_i\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir}_{\mathsf{T}}(x), [g])$ such that for every sequence $\{h_i\}_i^g \in CP(g)$ there exists $j \in N_g$ such that $|M_{j+1}| < h_j$, $M_1 = \mathsf{dir}_{\mathsf{T}}(x)$, and for all $d \in M_i - M_{i+1}$ there exists a successor y' of y with label $\mathsf{run}(y) = (xd, q', i)$. Since, $Dc(x) = \mathsf{dir}_{\mathsf{T}'}(x) \subseteq \mathsf{dir}_{\mathsf{T}}(x)$, we can construct the sequence $\{M_i'\}_i^{g+1}$ with $M_i' = M_i \cap Dc(x)$. Since $M_{g+1}' \subseteq \ldots \subseteq M_1'$, $M_1' = Dc(x) = \mathsf{dir}_{\mathsf{T}'}(x)$, and for every sequence $\{h_i\}_i^g \in CP(g)$ there exists $j \in N_g$ such that $|M_{j+1}'| \le |M_{j+1}| \le h_j$, we have $\{M_i'\}_i^{g+1} \in \mathsf{spart}(\mathsf{dir}_{\mathsf{T}'}(x), [g])$. Hence, for all $d \in M_i' - M_{i+1}' \subseteq M_i - M_{i+1}$ there exists a node $y' \in \mathsf{T_{pr}}$ such that $\mathsf{run}(y') = (xd, q', i)$, and such node exists in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ because $xd \in \mathsf{T}'$.

Since the run $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ is constructed from $\langle \mathsf{T_r}, \mathsf{run} \rangle$ by removing some nodes, every infinite branch of$\langle \mathsf{T_r}', \mathsf{run}' \rangle$ is an infinite accepting branch of $\langle \mathsf{T_r}, \mathsf{run} \rangle$. So, every infinite branch of $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ is accepting, and $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ is accepting.

Now it's a matter of redenominating the nodes, in order to have $\mathsf{T}' \subseteq [D-1]^*$.

Definition 20. (Satisfiability function) *A satisfiability function* $\mathsf{sat} : (H, \sigma) \in 2^{Q \times \mathbb{N}_{(b)}} \times \Sigma \mapsto \mathsf{sat}(H, \sigma) \in 2^{2^{D_b^\varepsilon \times Q}}$ *maps a set* H *and a label* σ *into a set of subset of* $D_b^\varepsilon \times Q$ *such that for all* $S \subseteq D_b^\varepsilon \times Q$ *it holds that* $S \in \mathsf{sat}(H, \sigma)$ *iff* $S \models \bigwedge_{(q,g)\in H} \delta(q, g, \sigma)$.

Definition 21. (Develop function) *A* develop function $\mathsf{dev} : (H,\sigma,d) \in 2^{Q\times\mathbb{N}_{(b)}} \times \Sigma \times \mathbb{N} \mapsto \mathsf{dev}(H,\sigma,d) \in 2^{2^{\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}}}$ *maps a set* H*, a label* σ*, and a number* d *into a set of subset of* $\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}$ *such that for all* $E \subseteq \mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}$ *it holds that* $E \in \mathsf{dev}(H,\sigma,d)$ *iff there exists* $S \in \mathsf{sat}(H,\sigma)$ *such that* $E \in \mathsf{exec}(S,\mathbb{N}_{(d)})$.

Definition 22. (Pair develop function) *A* pair develop function $\mathsf{pairdev}$: $(H,H',\sigma,d) \in (2^{Q\times\mathbb{N}_{(b)}})^2 \times \Sigma \times \mathbb{N} \mapsto \mathsf{pairdev}(H,H',\sigma,d) \in 2^{(2^{\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}})^2}$ *maps the two sets* H *and* H'*, a label* σ*, and a number* d *into a pair of sets of subset of* $\mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}$ *such that for all* $E,E' \subseteq \mathbb{N}_\varepsilon \times Q \times \mathbb{N}_{(b)}$ *it holds that* $(E,E') \in \mathsf{pairdev}(H,H',\sigma,d)$ *iff* $E' \subseteq E$*,* $E \in \mathsf{dev}(H,\sigma,d)$*, and if* $H' = \emptyset$ *then* $E' = E$ *otherwise* $E' \in \mathsf{dev}(H',\sigma,d)$.

Definition 23. (prefix function)
$pref : 2^{N_{D-1}\cup\{\varepsilon\}\times Q\times N_b} \times (N_{D-1}\cup\{\varepsilon\}) \to 2^{Q\times N_b}$ is a function that maps a pair (E,k) into a set $pref(E,k) = \{(q,d) \in 2^{Q\times N_b} \mid (k,q,d) \in E\}$.

Definition 24. (Map function)
$p : N_{D-1}\cup\{\varepsilon\} \to N_D$ is a function that maps a index $k \in N_{D-1}$ into another index $p(k)$, such that $p(k) = k$ if $k \in N_{D-1}$, and $p(\varepsilon) = D$.

Definition 25. (Extraction function)
For any $t \in N$, and $I \subseteq N^*$ we define the set $W_I(t) = \{x \in I \mid \exists y \in (N - \{t\})^*, z \in N^* \mid x = ytz\}$, and the function $ext_t : I \to I$ such that

1. *for all $x = ytz \in W_I(t)$ with $y \in (N-\{t\})^*$ we have $ext_t(x) = yz$,*
2. *for all $x \in I - W_I(t)$ we have $ext_t(x) = x$.*

Since for all $x \in I$ $|x| < +\infty$, for all $x \in I$ there exists $i_{x,t} \leq |x|$ such that $ext_t^{i_{x,t}}(x) = ext_t^{i_{x,t}+1}(x)$. This is because, by applying ext, eventually all occurences of t are removed.

So, we can define $ext_t^{+\infty} : I \to I$ such that for all $x \in I$ $ext_t^{+\infty}(x) = ext_t^{i_{x,t}}(x)$.

Proof. Here, we give the construction of the nondeterministic automaton $\mathcal{A}'$ from a partitioning alternating automaton $\mathcal{A}$. By using a non trivial proof, it can be shown that $\mathcal{L}(\mathcal{A}) \neq \emptyset$ iff $\mathcal{L}(\mathcal{A}') \neq \emptyset$. We postpone the details to an extended version. For a PABT $\mathcal{A} = \langle Q,\Sigma,b,\delta,q_0,g_0,F\rangle$ we construct an NBT $\mathcal{A}' = \langle Q',\Sigma,d',\delta',q'_0,F'\rangle$ as follows:

1. $Q' = (2^{Q\times\mathbb{N}_{(b)}})^2$;
2. $d' = n * b(b+1)/2$;
3. $q'_0 = (\{(q_0,g_0)\},\emptyset)$;
4. $F' = 2^{Q\times\mathbb{N}_{(b)}} \times \{\emptyset\}$;
5. $\delta' : Q' \times \Sigma \mapsto 2^{Q'^{(d'+1)}}$ is such that for all $H \subseteq Q \times \mathbb{N}_{(b)}$, $H' \subseteq H$ and $\sigma \in \Sigma$, we have:

$$\delta'((H,H'),\sigma) = \bigcup_{\substack{(E,E')\in\\ \mathsf{pairdev}(H,H',\sigma,d'-1)}} \{(\prod_{d=0}^{d'-1}(E_d, E'_d \setminus F)) \times (E_\varepsilon, E'_\varepsilon \setminus F)\}$$

with $E_d = pref(E,d)$.

(i) If $\mathcal{A}$ accepts an input tree, by lemma 16 $\mathcal{A}$ accepts an input tree $\langle \mathsf{T}, \mathsf{inp}\rangle$ with $\mathsf{T} \subseteq [D-1]^*$, and $D = |Q|\frac{b(b-1)}{2}$. We call $\langle \mathsf{T_r}, \mathsf{run}\rangle$ an accepting run of $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp}\rangle$.

Now we construct a new input tree $\langle \mathsf{T}', \mathsf{inp}'\rangle$ for $\mathcal{A}'$ such that

1. $\mathsf{T}' = [D]^*$.
2. for all $x \in \mathsf{T} \subseteq \mathsf{T}'$ holds $\mathsf{inp}'(x) = \mathsf{inp}(x)$.
3. for all $x \in [D-1]^* - \mathsf{T}$ holds $\mathsf{inp}'(x) = (\emptyset, \emptyset)$.
4. for all $x \in W_{\mathsf{T}'}(D)$ holds $\mathsf{inp}'(x) = \mathsf{inp}'(ext_D^{+\infty}(x))$,

We can see that inp is defined everywhere in T' since $[D]^* = [D-1]^* \cup W_{\mathsf{T}'}(D)$.

Now we will show that $\mathcal{A}'$ accepts $\langle \mathsf{T}', \mathsf{inp}'\rangle$, by constructing an accepting run $\langle \mathsf{T_r}', \mathsf{run}'\rangle$ of $\mathcal{A}'$ on $\langle \mathsf{T}', \mathsf{inp}'\rangle$ such that $\mathsf{T_r}' = \mathsf{T}'$. The run is constructed iteratively: at each step i we already constructed the first i levels of the run, and those levels satisfy the following inductive hypothesis.

1. For all $y \in \mathsf{T}'$ such that $ext_D^{+\infty}(y) \in \mathsf{T}$, $|y| \leq i$, and $\mathsf{run}'(y) = (H, H')$ we have that for all pairs $(q,d) \in H$ there exists a node $z \in \mathsf{T_r}$ such that $|z| = |y|$, and $\mathsf{run}(z) = (ext_D^{+\infty}(y), q, d)$.
2. For all $y \in \mathsf{T}'$ such that $ext_D^{+\infty}(y) \in \mathsf{T}' - \mathsf{T}$, and $|y| \leq i$ we have $\mathsf{run}'(y) = (\emptyset, \emptyset)$.
3. For every path π in $\langle \mathsf{T_r}', \mathsf{run}'\rangle$ such that
   (a) $|\pi| \leq i+1$,
   (b) for all $j \in N_{|\pi|-1}$ $ext_D^{+\infty}(\pi(j)) \in \mathsf{T}$,
   (c) for all $j \in N_{|\pi|-1}$ $\mathsf{run}'(\pi(j)) = (H_j, H_j')$,

   we have that for all $(q,d) \in H_{|\pi|-1}$ there exists a path π' in $\langle \mathsf{T_r}, \mathsf{run}\rangle$ such that $|\pi'| = |\pi|$, for all $j \in N_{|\pi'|-1}$ $\mathsf{run}(\pi'(j)) = (ext_D^{+\infty}(\pi(j)), q_j, d_j)$ with $(q_j, d_j) \in H_j$, and $(q_{|\pi|-1}, d_{|\pi|-1}) = (q,d)$.
4. For all $y \in \mathsf{T}'$ such that $ext_D^{+\infty}(y) \in \mathsf{T}$, $|y| \leq i$, and $\mathsf{run}'(y) = (H, H')$ we have that:
   (a) for all $(q,d) \in H'$ holds $(q,d) \notin F$,
   (b) for all $(q,d) \in H'$ there exists a node $z \in \mathsf{T_r}$ such that:
      i. $|z| = |y|$,
      ii. $\mathsf{run}(z) = (ext_D^{+\infty}(y), q, d)$,
      iii. does not exist a node $y' \in \mathsf{T}'$ such that:
         – y is a descendant of y'
         – $\mathsf{run}'(y') = (H_{y'}, \emptyset)$,
         – does not exist a node $y'' \in \mathsf{T}'$ such that:
            • y'' is a descendant of y',
            • y'' is an ancestor of y,
            • $\mathsf{run}'(y'') = (H_{y''}, \emptyset)$,
         – there exist $(q', d') \in H_{y'}$ and $z' \in \mathsf{T_r}$ such that:
            • $|z'| = |y'|$,
            • $\mathsf{run}(z') = (ext_D^{+\infty}(y'), q', d')$,
            • z is a descendant of z'
            • between z and z' there exists a node $z'' \in \mathsf{T_r}$ such that:
               ∗ z'' is a descendant of z',
               ∗ z'' is an ancestor of z,

   * $z'' \neq z'$,
   * $\mathsf{run}(z'') = (x, q'', d'')$ with $(q'', d'') \in F$.

At the beginning we construct only the level 0 of the run. This level has only the root ε such that

1. $\mathsf{run}'(\varepsilon) = (\{(q_0, g_0)\}, \{(q_0, g_0)\})$,
2. $ext_D^{+\infty}(\varepsilon) = \varepsilon \in \mathsf{T}$,
3. there exists $\varepsilon \in \mathsf{T_r}$ such that $\mathsf{run}(\varepsilon) = (\varepsilon, q_0, g_0)$.

So, we can easily see that the inductive hypothesis are satisfied at the beginning of step 1.

At each step i we construct the successor of the nodes of level $i-1$. Since the nodes $y_{i,j}$ of level i are $(D+1)^i$ in number, the step i is an iteration of $(D+1)^i$ substeps j in every one of which we construct the successors of the node $y_{i,j}$ of level $i-1$.

Now we show the costruction of the successors of the node y.

1. If $ext_D^{+\infty}(y) \in \mathsf{T}' - \mathsf{T}$, then $\mathsf{run}'(y) = (\emptyset, \emptyset)$, and for all $k \in N_D$ $ext_D^{+\infty}(yk) \in \mathsf{T}' - \mathsf{T}$. Hence, in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ every node yk must have a label $\mathsf{run}'(yk) = (\emptyset, \emptyset)$ in order to respect the inductive hypothesis. We can show that the labels of the successors of y are the ones required, it suffices to show that $\prod_{i=1}^{D+1}(\emptyset, \emptyset) \in \delta'((\emptyset, \emptyset), \mathsf{inp}(y))$. Since $\emptyset \in sat(\emptyset, \mathsf{inp}(y))$, we have $\emptyset \in dev(\emptyset, \mathsf{inp}(y), D) = \emptyset$, so, $(\emptyset, \emptyset) \in pairdev(\emptyset, \emptyset, \mathsf{inp}(y), D)$, and $\pi_{i=1}^{D+1}(\emptyset, \emptyset) \in \delta'((\emptyset, \emptyset), \mathsf{inp}(y))$.
2. If $ext_D^{+\infty}(y) \in \mathsf{T}$, and $\mathsf{run}'(y) = (H, H')$, then, for all pairs $(q, d) \in H$ there exists a node $z_{q,d} \in \mathsf{T_r}$ such that $|z_{q,d}| = |y|$, and $\mathsf{run}(z_{q,d}) = (ext_D^{+\infty}(y), q, d)$.
   Precisely we can consider the path π in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ such that $|\pi| = i+1$, and $\pi(i) = y$. By inductive hipothesis 3, for all $(q, d) \in H$ there exists a path $\pi'_{q,d}$ in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ such that $|\pi'| = i+1$ and for all $j \in N_i$ $\mathsf{run}(\pi'_{q,d}(j)) = (ext_D^{+\infty}(\pi(j)), q_j, d_j)$. So, we can choose $z_{q,d} = \pi'_{q,d}(i)$.
   Hence, for all $(q, d) \in H$ there exists a sodisfacibility set $S_{q,d} \models \delta(q, d, \mathsf{inp}(ext_D^{+\infty}(y)))$ choosen during the construction of the successors of $z_{q,d}$ in the run $\langle \mathsf{T_r}, \mathsf{run} \rangle$. So, we have $S' = \cup_{(q,d) \in H} S_{q,d} \models \wedge_{(q,d) \in H} \delta(q, d, \mathsf{inp}_D^{+\infty}(y))$, and $S' \in sat(H, \mathsf{inp}(ext_D^{+\infty}(y)))$. According to the construction of S' for every sequence of sets $e_{q,d} \in \mathsf{exec}(S_{q,d}, N_D)$ dependant on $(q, d) \in H$ we have $\cup_{(q,d) \in H} e_{q,d} \in \mathsf{exec}(S', N_D) \subseteq dev(H, \mathsf{inp}(ext_D^{+\infty}(y))), D)$. Precisely for all $(q, d) \in H$ there exists a set $E_{q,d} \in \mathsf{exec}(S_{q,d}, N_D)$, such that for all $(k, q', d') \in E_{q,d}$ there exists a successor z' of $z_{q,d}$ with label $\mathsf{run}(z') = (ext_D^{+\infty}(yp(k)), q', d')$ with p the map function of definition 24.
   Hence $E = \cup_{(q,d) \in H} E_{q,d} \in dev(H, \mathsf{inp}(ext_D^{+\infty}(y))), D)$.
   In the same way one can show that $E' = \cup_{(q,d) \in H'} E_{q,d} \in dev(H', \mathsf{inp}(ext_D^{+\infty}(y))), D)$.
   Moreover, $H' \subseteq H$, so $E' \subseteq E$, and:
   (a) if $H' \neq \emptyset$ we have $(E, E') \in paidev(H, H', \mathsf{inp}(ext_D^{+\infty}(y))), D)$.
   (b) If $H' = \emptyset$ we must set $E' = E$ and we have $(E, E') \in paidev(H, H', \mathsf{inp}(ext_D^{+\infty}(y))), D)$.
   Hence, $\prod_{i=0}^{D-1}(pref(E, i), pref(E', i) - F) \times (pref(E, \varepsilon), pref(E', \varepsilon) - F)$ $\in \delta'((H, H'), \mathsf{inp}(ext_D^{+\infty}(y))$.
   So we can construct the successor of y using the above element of the set $\delta'((H, H'), \mathsf{inp}(ext_D^{+\infty}(y))$. In this way every successor $yp(k)$ has label $\mathsf{run}'(yp(k)) = (pref(E, k), pref(E', k) - F) = (H_k, H'_k)$.

Now we will show that the inductive hypothesis holds after step i. However, the properties can only be voided by the nodes added during step i, so we will simply show that the properties hold only when at least one of the partecipant node is a successor of some node y of level $i-1$.

(a) For all $yp(k) \in \mathsf{succ}_{\mathsf{T}'}(y)$ such that $ext_D^{+\infty}(yp(k)) \in \mathsf{T}$, $|yp(k)| = i+1$, and $\mathsf{run}'(yp(k)) = (H_k, H'_k)$ we have that for all pairs $(q,d) \in H_k = pref(E,k) = \{(q',d') \in Q \times N_b \mid (k,q',d') \in \cup_{(q,d)\in H} E_{q,d}\}$ there exists a pair (q',d') such that $(k,q,d) \in E_{q',d'}$. So, there exists a node $z' \in \mathsf{succ}_{\mathsf{T_r}}(z_{q',d'})$ such that $|z'| = |yp(k)|$, and
$\mathsf{run}(z') = (ext_D^{+\infty}(yp(k)), q, d)$.

(b) For all $yp(k) \in \mathsf{succ}_{\mathsf{T}'}(y)$, such that $ext_D^{+\infty}(yp(k)) \in \mathsf{T}' - \mathsf{T}$, and $|yp(k)| = i+1$ we have that $pref(E,k) = \emptyset$. This is because $pref(E,k) = \{(q',d') \in Q \times N_b \mid (k,q',d') \in \cup_{(q,d)\in H} E_{q,d}\}$ is empty, since, in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ y has no successor in the direction k. So, $\mathsf{run}'(yp(k)) = (\emptyset, \emptyset)$.

(c) For every path π'' in $\langle {\mathsf{T_r}}', \mathsf{run}' \rangle$ such that
   i. $|\pi''| = i+2$
   ii. for all $j \in N_{i+1}$ $ext_D^{+\infty}(\pi''(j)) \in \mathsf{T}$,
   iii. for all $j \in N_{i+1}$ $\mathsf{run}'(\pi''(j)) = (H_j, H'_j)$, $ext_D^{+\infty}(\pi''(j)) \in \mathsf{T}$,
   iv. there exists a direction k such that $\pi''(i+1) = yp(k)$

   we have that the path $\pi = \pi''_{\leq i}$ satisfy the inductive hypothesis 3.
   Hence, for all $(q,d) \in H_i$ there exists a path $\pi'_{q,d}$ such that $|\pi'_{q,d}| = i+1$, and for all $j \in N_i$ $\mathsf{run}(\pi'_{q,d}(j)) = (ext_D^{+\infty}(\pi(j)), q_j, d_j)$ with $(q_i, d_i) = (q,d)$.
   Let's remember that $\mathsf{run}'(yp(k)) = (H_{i+1}, H'_{i+1})$, since, for all $(q',d') \in H_{i+1}$ there exists $(q,d) \in H_i$ and there exists $z' \in \mathsf{succ}_{\mathsf{T_r}}(z_{q,d})$ such that $\mathsf{run}(z') = (ext_D^{+\infty}(yp(k)), q', d')$, then for all $(q',d') \in H_{i+1}$ we can construct the path $\pi'''_{q',d'}$ such that for all $j \in N_i$ $\pi'''_{q',d'}(j) = \pi'_{q,d}(j)$ e $\pi'''_{q',d'}(i+1) = z'$.
   So, for every above path π'' for all $(q',d') \in H_{i+1}$ there exists a path $\pi'''_{q',d'}$ such that $\mathsf{run}(\pi'''_{q',d'}(j)) = (ext_D^{+\infty}(\pi''(j)), q_j, d_j)$ with $(q_j, d_j) \in H_j$, and $(q_{i+1}, d_{i+1}) = (q',d')$.

(d) For all $yp(k) \in \mathsf{succ}_{\mathsf{T}'}(y)$ such that $ext_D^{+\infty}(yp(k)) \in \mathsf{T}$, $|yp(k)| = i+1$, and $\mathsf{run}'(yp(k)) = (H_k, H'_k)$ we have that for all pairs $(q,d) \in H'_k = pref(E',k) - F = \{(q',d') \in Q \times N_b \mid (k,q',d') \in \cup_{(q,d)\in H'} E'_{q,d}\} - F$ there exists a pair (q',d') such that $(k,q,d) \in E'_{q',d'}$.
   Hence, $(q,d) \notin F$, and there exists a node $z' \in \mathsf{succ}_{\mathsf{T_r}}(z_{q',d'})$ such that $|z'| = |yp(k)|$, and $\mathsf{run}(z') = (ext_D^{+\infty}(yp(k)), q, d)$.
   At this point we need to show the inductive property $4-b-iii$ for the node z'.
   i. If $H' = \emptyset$, the only node y' that can void the property with its existence can be only y. At this point, for all $(q',d') \in H$ the only node z'' that can void the property is a node $z_{q',d'}$ such that $z' \in \mathsf{succ}_{\mathsf{T_r}}(z_{q',d'})$. At this point between $z_{q',d'}$ e z' the only node z'' that can void the property is z' with $\mathsf{run}(z') = (yp(k), q, d)$. However, by inductive hypothesis we know that $(q,d) \notin F$. So, the property cannot be voided.
   ii. If $H' \neq \emptyset$, the only node y' that can void the property with its existence is the nearest ancestor of y such that $\mathsf{run}'(y') = (H_{y'}, \emptyset)$.

Let's take an element $(q',d') \in H_{y'}$, and let $z'' \in \mathsf{T_r}$ be a node such that $|z''| = |y'|$, $\mathsf{run}(z'') = (ext_D^{+\infty}(y'), q', d')$, and z'' is an ancestor of z'. Let's call $z_{q'',d''}$ the father of z'.
In order to void property $4-b-iii$, between z'', and z' we can only take nodes z''' between z'' e $z_{q'',d''}$ or the node $z''' = z'$.
However, by induction hypothesis all nodes z''' between z'' and $z_{q'',d''}$ have label $\mathsf{run}(z''') = (x, q''', d''')$ with $(q''', d''') \notin F$, so they cannot void the property.
Moreover, if we choose $z''' = z'$, $\mathsf{run}(z') = (yp(k), q, d)$, and, by induction hypothesis, we know that $(q,d) \notin F$. So the property cannot be void neither with $z''' = z'$.

Now we show that the constructed run $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ is accepting. Let's suppose by contradiction that $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ is not accepting, then there exists a not accepting infinite branch π and a number $M \in N$ such that for all $j > M$ $\mathsf{run}(\pi(j)) = (H_j, H'_j)$ with $H'_j \neq \emptyset$, and $\mathsf{run}(\pi(M)) = (H_M, \emptyset)$.

For all $i > M$ we can take in account the paths π_i which are prefix of π until step i.

Then by the induction property 3, for every path π_i there exists a set P_i of paths π'_i in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ such that for all $j \in N_i$ $\mathsf{run}(\pi'_i(j)) = (ext_D^{+\infty}(\pi(j)), q_j, d_j)$ with $(q_j, d_j) \in H_j$.

Moreover for all nodes $\pi(i)$ such that $i > M$, the only node y' that can void the induction property 4 on the path π until $\pi(i)$ is the node $\pi(M)$.

Hence, there does not exist a pair $(q,d) \in H_M$ and a node $z' \in \mathsf{T_r}$ such that

1. $|z'| = |\pi(M)|$,
2. $\mathsf{run}(z') = (ext_D^{+\infty}(\pi(M)), q, d)$,
3. z is a descendant of z'
4. between z e z' there exists a node $z'' \in \mathsf{T_r}$ such that:
   (a) z'' is a descendant of z',
   (b) z'' is an ancestor of z,
   (c) $z'' \neq z'$
   (d) $\mathsf{run}(z'') = (x, q'', d'')$ with $(q'', d'') \in F$.

This implies that for all $i > M$ and for all $j > M$ $\mathsf{run}(\pi'_i(j)) = (ext_D^{+\infty}(\pi(i)), q_i, d_i)$ with $(q_i, d_i) \notin F$.

Since, on the first $i-1$ nodes π'_i satisfies the property that for all $j \in N_{i-1}$ $\mathsf{run}(\pi'_i(j)) = (ext_D^{+\infty}(\pi(j)), q_j, d_j)$ with $(q_j, d_j) \in H_j$, we have that the path π'_{i-1} obtained from π'_i by removing its last node is in the set of paths P_{i-1}. So every path $\pi'_i \in P_i$ is obtained from a path $\pi'_{i-1} \in P_{i-1}$ by adding a node at the end.

This property shows that for $i \rightarrow +\infty$ we can construct an infinite path π^{+infty} in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ such that every finite prefix of j steps is a path of P_j.

Since for all finite path $\pi'_i \in P_i$ with $i > M$, and for all $j > M$ $\mathsf{run}(\pi'_i(j)) = (ext_D^{+\infty}(\pi(i)), q_i, d_i)$ with $(q_i, d_i) \notin F$, we have that, starting at step M, π^{+infty} does not have labels with states in F.

So, $\langle \mathsf{T_r}, \mathsf{run} \rangle$ is not accepting, and this contradicts the hypothesis of the theorem.

(ii) Let's suppose that $\mathcal{A}'$ accepts the input tree $\langle \mathsf{T}', \mathsf{inp}' \rangle$ such that $\mathsf{T}' = [D]^*$, and let's call $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ an accepting run of $\mathcal{A}'$ on $\langle \mathsf{T}', \mathsf{inp}' \rangle$ with $\mathsf{T_r}' = \mathsf{T}'$.

We can show that $\mathcal{A}$ accepts the input tree $\langle \mathsf{T}, \mathsf{inp} \rangle$ such that $\mathsf{T} = [D-1]^*$, and for all $x \in \mathsf{T}$ $\mathsf{inp}(x) = \mathsf{inp}'(x)$, by iteratively constructing a run $\langle \mathsf{T_r}, \mathsf{run} \rangle$ of $\mathcal{A}$ on $\langle \mathsf{T}, \mathsf{inp} \rangle$.

At each step i of the iteration we already constructed the first i levels of the run, and those levels satisfy the following inductive hypothesis.

1. For all nodes $z \in \mathsf{T_r}$ such that $|z| \leq i$, and $\mathsf{run}(z) = (x,q,d)$, we have that there exists a node $y \in \mathsf{T}'$ such that $|y| = |z|$, $exp_D^{+\infty}(y) = x$, and $\mathsf{run}'(y) = (H,H')$ con $(q,d) \in H$.
2. For every path π in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ such that:
   (a) $|\pi| \leq i+1$,
   (b) for all $j \in N_i$ $\mathsf{run}(\pi(j)) = (x_j, q_j, d_j)$,

   there exists a path π' in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ such that:
   (a) $|\pi'| \leq i+1$,
   (b) for all $j \in N_i$ $ext_D^{+\infty}(\pi'(j)) = x_j$,
   (c) for all $j \in N_i$ $\mathsf{run}'(\pi'(j)) = (H_j, H'_j)$ with $(q_j,d_j) \in H_j$,
   (d) if $(q_j,d_j) \in H'_j$, and $(q_{j+1},d_{j+1}) \notin F$ then $(q_{j+1},d_{j+1}) \in H'_{j+1}$.

At the beginning we construct only the level 0 of the tree. This level has only the root ε such that

1. $\mathsf{run}'(\varepsilon) = (\varepsilon, q_0, g_0)$,
2. $ext_D^{+\infty}(\varepsilon) = \varepsilon \in \mathsf{T}$,
3. there exists $\varepsilon \in \mathsf{T}'$ such that $\mathsf{run}'(\varepsilon) = (\{q_0,g_0\},\{q_0,g_0\})$.

So, we can easily see that the inductive hypothesis are satisfied at the beginning of step 1.

At each step i we construct the successor of the nodes of level $i-1$, by using an inner iteration of substeps j in every one of which we construct the successors of the node $z_{i,j}$ of level $i-1$.

Now we show the costruction of the successors of the node z.

We set $\mathsf{run}(z) = (x,q,d)$ with $|z| = i$, then by inductive hypothesis there exists a node $y \in \mathsf{T}'$ such that $|y| = i$, $exp_D^{+\infty}(y) = x \in \mathsf{T}$, and $\mathsf{run}'(y) = (H,H')$ with $(q,d) \in H$.

Moreover there exists a path π in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ such that $|\pi| = i+1$, $\pi(i) = z$ and for all $j \in N_i$ $\mathsf{run}(\pi(j)) = (x_j,q_j,d_j)$.

So by inductive hypothesis there exists a path π' in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ such that $|\pi'| = i+1$, $exp_D^{+\infty}(\pi'(j)) = x_j \in \mathsf{T}$, and $\mathsf{run}'(\pi'(j)) = (H_j,H'_j)$ con $(q_j,d_j) \in H_j$. So we can choose $y = \pi'(i)$.

Hence, there exists $S \in sat(H, \mathsf{inp}(x)) = sat(H, \mathsf{inp}'(y))$ such that there exists $E \in \mathsf{exec}(S, N_{D-1} \cup \{\varepsilon\})$ such that there exists $E' \subseteq E$ such that

1. $(E,E') \in pairdev(H,H',\mathsf{inp}(x)) = pairdev(H,H',\mathsf{inp}'(y))$,
2. $\prod_{i=0}^{D-1}(pref(E,i), pref(E',i)-F) \times (pref(E,\varepsilon), pref(E',\varepsilon)-F) \in \delta'((H,H'),\mathsf{inp}(x)) = \delta'((H,H'),\mathsf{inp}'(y))$
3. $\prod_{i=0}^{D-1}(pref(E,i), pref(E',i)-F) \times (pref(E,\varepsilon), pref(E',\varepsilon)-F)$ is choosen in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ for the construction of the successor of y.

Since, $S \models \wedge_{(q',d') \in H} \delta(q',d',\mathsf{inp}(x))$, and precisely $S \models \delta(q,d,\mathsf{inp}(x))$, S is a good sodisfacibility set for the construction of the successor of z.

Besides, if $H' \neq \emptyset$ then there exists $S' \in sat(H',\mathsf{inp}(x))$ such that $E' \in \mathsf{exec}(S', N_{D-1} \cup \{\varepsilon\})$ with $E' \subseteq E$. Since $S' \models \wedge_{(q',d') \in H'} \delta(q',d',\mathsf{inp}(x))$, S' is a good sodisfacibility set for the construction of the successor of z when $(q,d) \in H'$.

So there are two possible situations.

1. If $(q,d) \notin H'$, we use the set $E \in \mathsf{exec}(S, N_{D-1} \cup \{\varepsilon\})$, for the construction of the successors of z.
   For all $(k,q',d') \in E$ there exists a successor z' of z such that $\mathsf{run}(z') = (xk,q',d')$, $|yp(k)| = |z'|$, $\mathsf{run}'(yp(k)) = (pref(E,k), pref(E',k) - F)$ con $(q',d') \in pref(E,k)$.
   So, by construction, the iterative hypothesis 1 is satisfied on such successors of z.
2. If $(q,d) \in H'$, we use the set $E' \in \mathsf{exec}(S', N_{D-1} \cup \{\varepsilon\})$, for the construction of the successor of z.
   For all $(k,q',d') \in E'$ there exists a successor z' of z such that $\mathsf{run}(z') = (xk,q',d')$, $|yp(k)| = |z'|$, and $\mathsf{run}'(yp(k)) = (pref(E,k), pref(E',k) - F)$ con $(q',d') \in pref(E',k) \subseteq pref(E,k)$
   Since $(q',d') \notin F$, then $(q',d') \in pref(E',k) - F$.
   So, by construction, the iterative hypothesis 1 is satisfied on such successors of z.

At the end of the step i the iterative hypothesis is satisfied on all nodes of the run constructed so far.

Now we will show that the inductive hypothesis 2 holds after step i. However, the property can only be voided by the nodes added during step i, so we will simply show that the properties hold only when at least one of the partecipant node is a successor of some node z of level $i-1$.

For all paths π'' in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ such that

1. $|\pi''| = i+2$,
2. $\mathsf{run}(\pi''(i+1)) = (x_i k, q', d')$,

we have that

1. π is a prefix of length $i+1$ of π'',
2. there exist $k \in N_{D-1} \cup \{\varepsilon\}$ and $(q',d') \notin F$ such that
   (a) $(k,q',d') \in E$,
   (b) $|yp(k)| = |z'|$,
   (c) $\mathsf{run}'(yp(k)) = (pref(E,k), pref(E',k) - F)$
   (d) if $(q,d) \in H$, and $(q',d') \notin F$, then $(q',d') \in pref(E',k) - F$.

So, we can construct the path π''' in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ of length $i+2$ such that for all $j \in N_i$ $\pi'''(j) = \pi'(j)$, and $\pi'''(i+1) = \pi'(i)p(k)$. Hence, path π'' satisfies the property 2.

Now we show that the constructed run $\langle \mathsf{T_r}, \mathsf{run} \rangle$ is accepting.

Let's suppose, by contradiction, that $\langle \mathsf{T_r}, \mathsf{run} \rangle$ is not accepting, then there exist a not accepting infinite branch π and a number $M \in N$ such that for all $i > M$ $\mathsf{run}(\pi(i)) = (x_i, q_i, d_i)$, with $(q_i,d_i) \notin F$.

We can call π_i the prefixes of π of length i. Then by the inductive property 2 there exists a set P_i of paths π'_i in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ such that for all $j \in N_i$

1. $|\pi'_i(j)| = |\pi(j)|$,
2. $ext_D^{+\infty}(\pi'_i(j)) = x_j$,
3. there exists H_j, H'_j such that $\mathsf{run}'(\pi'_i(j)) = (H_j, H'_j)$, and $(q_j, d_j) \in H_j$.
4. if $(q_j, d_j) \in H'_j$, and $(q_{j+1}, d_{j+1}) \notin F$ then $(q_{j+1}, d_{j+1}) \in H'_{j+1}$.

Since, on the first $i-1$ nodes π'_i satisfies the property that for all $j \in N_{i-1}$ $|\pi'_i(j)| = |\pi(j)|$, $ext_D^{+\infty}(\pi'_i(j)) = x_j$, and $\mathsf{run}'(\pi'_i(j)) = (H_j, H'_j)$ with $(q_j, d_j) \in H_j$, we have that the path π'_{i-1} obtained from π'_i by removing its last node is in the set of paths P_{i-1}. So every path $\pi'_i \in P_i$ is obtained from a path $\pi'_{i-1} \in P_{i-1}$ by adding a node at the end.

This property shows that for $i \to +\infty$ we can construct an infinite path π^{+infty} in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ such that every finite prefix of j steps is a path of P_j.

Hence, for all $i \in N$ $|\pi^{+\infty}(i)| = |\pi(i)|$, $ext_D^{+\infty}(\pi^{+\infty}(i)) = x_i$, and $\mathsf{run}'(\pi^{+\infty}(i)) = (H_i, H'_i)$ with $(q_i, d_i) \in H_i$.

Let r be an index such that $r > M$, and $H'_r = \emptyset$, such index must exists because $\pi^{+\infty}$ is an accepting branch in $\langle \mathsf{T_r}', \mathsf{run}' \rangle$. So, by construction of *pairdev*, we have $H'_{r+1} = H_{r+1}$, and, hence, we have $(q_{r+1}, d_{r+1}) \in H_{r+1}$. At last, by inductive hypothesis, we know that for all $j \in N$, if $(q_j, d_j) \in H'_j$, and $(q_{j+1}, d_{j+1}) \notin F$, then $(q_{j+1}, d_{j+1}) \in H'_{j+1}$.

Hence, by induction, we have that for all $i > r$ $(q_i, d_i) \in H'_i$, so, $H'_i \neq \emptyset$.

Then, the infinite branch π^{+infty} of $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ is not accepting, but this contradicts the hypothesis of the theorem.

Proof. (Resume of the proof)

In another lemma we showed that if $\mathcal{A}$ accepts an input tree then it accepts an input tree $\langle \mathsf{T}, \mathsf{inp} \rangle$ such that $\mathsf{T} \subseteq [D-1]^*$, where $D = |Q| \frac{b(b-1)}{2}$. So, in $\langle \mathsf{T}, \mathsf{inp} \rangle$ there are only D directions we must take care of, and, in a run of $\mathcal{A}$ we use always one more direction ε when we construct the successor of node.

In order to show the theorem we construct an *NBT* $\mathcal{A}'$ that accepts only input trees $\langle \mathsf{T}', \mathsf{inp}' \rangle$ such that $\mathsf{T}' = [D]^*$, with runs $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ such that $\mathsf{T_r}' = [D]^*$ So, we have enough directions to keep memory of what happens in a run of $\mathcal{A}$.

So if $\mathcal{A}$ accepts $\langle \mathsf{T}, \mathsf{inp} \rangle$ with a run $\langle \mathsf{T_r}, \mathsf{run} \rangle$, we can construct an input tree $\langle \mathsf{T}', \mathsf{inp}' \rangle$ that can be accepted by $\mathcal{A}'$ with a run $\langle \mathsf{T_r}', \mathsf{run}' \rangle$

In $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ a node $y = y_0 \dots y_n \in [D]^*$ is labeled with a pair $(H_y, H'_y) \in (2^{Q \times N_b})^2$. The set H remembers all the states of the nodes in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ that develeps along the directions $y_0, \dots, y_n$. Precisely $H_{y_0} = \{(q_0, g_0)\}$ remembers only the initial state of the root of $\langle \mathsf{T_r}, \mathsf{run} \rangle$. Then $H_{y_0 y_1}$ remembers only the states that develop along direction y_1 startiting from the root in $\langle \mathsf{T_r}, \mathsf{run} \rangle$, i.e., $H_{y_0 y_1}$ contains the states (q, d) of the successor of the root with label (y_1, q, d). Then $H_{y_0 y_1 y_2}$ remembers only the states that develop along direction y_2 starting from any node of level 1 in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ that had a state (y_1, q, d), i.e., $H_{y_0 y_1 y_2}$ contains the states (q, d) of the successor of the root with label $(y_1 y_2, q, d)$.

The set H' remembers all the states of the nodes in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ that develops along the directions $y_0, \dots, y_n$, except those states that are either accepting or come from other accepting states. Precisely $H'_{y_0} = \{(q_0, g_0)\} - F$ remembers only the initial state of the root of $\langle \mathsf{T_r}, \mathsf{run} \rangle$ if it is not accepting. Then $H'_{y_0 y_1}$ remembers only the not accepting states that develop along direction y_1 startiting from the root in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ if the the root had a not accepting state. Then $H'_{y_0 y_1 y_2}$ remembers only the states that develop along direction y_2 starting from any node of level 1 in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ that had a state (y_1, q, d) with $(q,d) \in H'_{y_0 y_1}$. It is obvious that along the directions $y_0, \ldots, y_n$ the set $H'_{y_0 \ldots y_k}$ can eventually become empty, at this point we always set $H'_{y_0 \ldots y_{k+1}} = H_{y_0 \ldots y_{k+1}}$.

The only problem with the direction is that the direction ε in $\langle \mathsf{T_r}, \mathsf{run} \rangle$ becomes direction D in the run $\langle \mathsf{T_r}', \mathsf{run}' \rangle$.

The set H' is used in order to check the acceptance of the run $\langle \mathsf{T_r}, \mathsf{run} \rangle$, by checking that on every infinite branch of the run, H' becomes empty infinitely often. So on every branch of $\langle \mathsf{T_r}, \mathsf{run} \rangle$ an accepting state appears infinitely often, iff on every infinite sequence of direction of N^w the paths of $\langle \mathsf{T_r}, \mathsf{run} \rangle$ that develops along those directions visits an accepting state infinitely often. So, this happens iff on every infinite sequence of directions of N^w on the paths of $\langle \mathsf{T_r}', \mathsf{run}' \rangle$ the set H' becomes empty infinitely often.

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