# Balance Games on Colored Graphs

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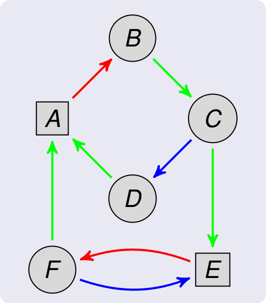
# The problem (I)




- Edge-colored finite 2-player arenas.
- Find a strategy of player 1 (circle) such that all resulting infinite paths have colors satisfying given frequency-related goals.
- Three goals:
  - balance;
  - bounded;
  - frequency-$f$.

# The problem (II)

Balance property: a path is balanced iff all colors appear with the same asymptotic frequency.

Bounded property: a path is bounded iff there is a constant that bounds the difference between the number of occurrences of any two given colors.

Frequency-$f$ property: a path has frequency-$f$ iff each color $c$ has frequency $f(c)$ assigned by the function $f$.

# Our motivation (I)

- Arenas represent concurrent systems of interactions between two different entities (e.g. schedulers, etc.).
- Colors represent the progress of interaction (e.g., jobs, etc.).
- We want a fair scheduling of jobs.

- Important problem not so much investigated.
- It is possible to use several notions of fairness (e.g. strong, weak, etc.).
- The basic and well-known notion: all colors appear infinitely often.

# Our motivation (II)

In a previous work we developed the concepts of balanced and bounded paths as a quantitative refinement of classical notions of fairness.

With this concept, we can formally state a quantitative comparison between the occurrences of competing events.

In this work, we further generalize these concepts and apply them to the contest of concurrent system verification through a game-theoretic framework.

# Two applications

Player 1 (circle): Main scheduler. Player 0 (square): Auxiliary schedulers.

- Main scheduler decides which macro-operation has to be executed.
- Auxiliary schedulers select some sub-operation of the macro-operation.

Player 1 (circle): Scheduler. Player 0 (square): Internal nondeterminism.

- Schedulers decides which jobs has to be executed.
- The choice of internal task for each job is nondeterministic.

The game model allows us to check the ability of schedulers to force a fair progress of actions, where by fair we mean one of our three frequency-related goals (balance, bounded, frequency-$f$).

# Outline

# Notation

- Let $\mathcal{C}$ be a finite set of colors $\{1, 2, \dots, k\}$.
- With $\rho = \rho_0\rho_1 \dots$ we denote a sequence of colors derived by a path.
- $\rho^n$ is the finite prefix of $\rho$ with length $n$.
- $|\rho|_c$ is the number of occurrences of $c \in \mathcal{C}$ in $\rho$.
- For simplicity, we identify paths in the arena and sequences of colors.

# Notions

Balance property: a path is balanced iff all colors appear with the same asymptotic frequency, i.e., $\forall c \in \mathcal{C}.\ \lim_{n\to\infty} \frac{|\rho^n|_c}{n} = \frac{1}{k}$.

Bounded property: a path is bounded iff there is a constant that bounds the difference between the number of occurrences of any two given colors, i.e., $\exists m \in \mathbb{N}.\ \forall c_1, c_2 \in \mathcal{C}.\ \forall n \in \mathbb{N}.\ |\rho^n|_{c_1} - |\rho^n|_{c_2} \leq m$.

Frequency-$f$ property: a path has frequency-$f$ iff each color $c$ has frequency $f(c)$ assigned by the function $f$, i.e., $\forall c \in \mathcal{C}.\ \lim_{n\to\infty} \frac{|\rho^n|_c}{n} = f(c)$, where $\sum_{c\in\mathcal{C}} f(c) = 1$.

# Decision problems

Given a finite arena, determine the existence of a strategy of player 1 such that, independently of the strategy of player 0, the resulting path

1. is balanced (balance game);
2. is bounded (bounded game);
3. has frequency-$f$ (frequency game).

# Computational complexity

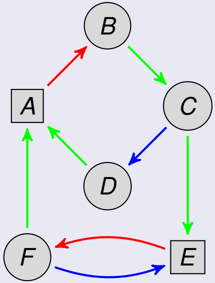
| | 1-player | 2-players |
|---|---|---|
| Balance game | PTIME[1] | CoNPTIME-COMPLETE[2] |
| Bounded game | PTIME[1] | CoNPTIME-COMPLETE[2] |
| Frequency game | PTIME[2] | CoNPTIME-COMPLETE[2] |
| Finite balance game | NPTIME-COMPLETE[1] | PSPACE-HARD |

Table: Computational complexity of 1-player / 2-players games.

[1] A. Bianco, M. Faella, F. Mogavero, A. Murano. Balanced Paths in Colored Graphs, MFCS'09.

[2] A. Bianco, M. Faella, F. Mogavero, A. Murano. Colored Games with Frequency Goals, Submitted.

# Winning strategy for player 1




- Player 1 controls nodes $B$, $C$, $D$, and $F$.
- His goal is to construct a balanced path.
- Player 0 has no choice.
- A possible balanced path is

$$\rho = \prod_{i=0}^{\infty} A \cdot B \cdot C \cdot (D \cdot A \cdot B \cdot C)^{i} \cdot (E \cdot F)^{(i+1)}.$$

- The $i$-th block is $6i+5$ edges long: $2i+1$ red's, $2i$ blue's, and $2i+3$ green's.
- Player 1 has a winning strategy with infinite memory (no finite-memory strategy exists).

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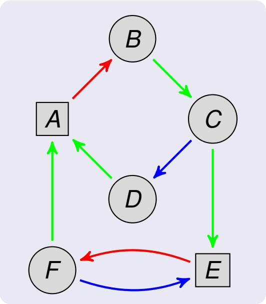


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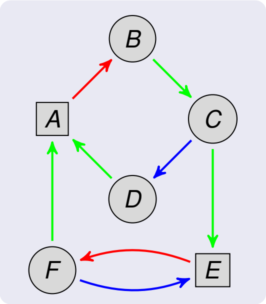


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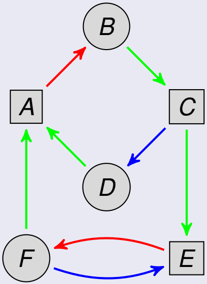
# Winning strategy for player 0




- Player 0 controls nodes $A$, $C$, and $E$.
- His goal is to avoid a balanced path.
- Player 1 has choice only on $F$.
- A winning memoryless strategy for player 0 is to choose everytime the edge from $C$ to $D$.
- The only resulting path

$$\rho = (A \cdot B \cdot C \cdot D)^{\omega}$$

is not balanced.

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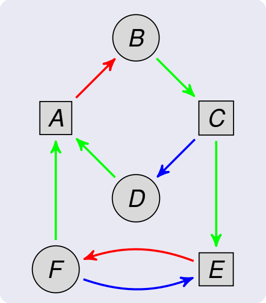


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# Outline

# Definitions

A preference relation $\leq$ is a total preorder (i.e., reflexive and transitive relation) on the infinite sequences of colors.

Two strategies, $\sigma$ for player 0 and $\tau$ for player 1, are $\leq$-optimal iff, for all strategies $\sigma'$ of player 0 and $\tau'$ of player 1, it holds that

$$\mathrm{path}(\sigma', \tau) \leq \mathrm{path}(\sigma, \tau) \leq \mathrm{path}(\sigma, \tau').$$

# Properties

We generalize the concepts of monotonicity and selectivity of Zielonka[1]. Informally, we have the following.

- Strongly monotone: A preference relation is strongly monotone iff, at each moment during a play, a player can choose an optimal extension of the play independently from the preceding finite play (i.e., the choice is history-invariant).
- Strongly selective: A preference relation is strongly selective iff a player cannot improve his payoff by switching between different behaviors rather than following a fixed one.

[1] H. Gimbert, W. Zielonka. Games where you can play optimally without any memory, CONCUR'05.

# One memoryless strategy

**Theorem**

*Let $\leq$ be a strongly monotone and strongly selective preference relation.*

*Assume that, for all $k$-colored games $G$ controlled by player 1, there are two $\leq$-optimal strategies for both players.*

*Then, for all $k$-colored games $G'$, there exist also two $\leq$-optimal strategies for both players such that the strategy of player 0 is memoryless.*

# Outline

# Preference relations for our goals

**Theorem**

*Let $(P_0, P_1)$ be a partition of $\mathcal{C}^\omega$ for which in $P_1$ there are all and only the paths that are winning for player 1 (w.r.t. one of our winning conditions).*

*Let $\leq$ be the preference relation such that $x \leq x'$ iff either $x' \in P_0$ or $x, x' \in P_1$.*

*Then, $\leq$ is strongly monotone and strongly selective.*

By using the previous theorem we obtain that, for all colored game with preference relation $\leq$, there exist two $\leq$-optimal strategies for both players such that the strategy for player 0 is memoryless.

# The algorithm (I)

As resolutive algorithm, we use the following idea.

- If player 0 has a winning strategy, then he has a memoryless winning strategy $\sigma$.
- Hence, we decide whether there exists a winning strategy $\tau$ for player 1 by simply checking whether all memoryless strategies $\sigma$ for player 0 are non-winning.

# The algorithm (II)

To do this,

- for each memoryless strategy $\sigma$ of player 0, we prune the game arena $G$ in accordance with $\sigma$, obtaining the new arena $G_\sigma$,
- then, we check whether, on the resulting subarena $G_\sigma$, there exists a path $\rho$ satisfying the winning condition of the game (we use the algorithm for one-player game).

Such a path $\rho$ does not exist iff the strategy for player 0 is winning.

In the end, by guessing which memoryless strategy for player 0 is winning, we obtain a CoNPTime algorithm that determines whether or not there exists a winning strategy for player 1.

# Boolean validity

For the lower bound, we use a reduction from the validity problem.

Given a boolean formulas $\varphi$ in CNF with $n$ clauses, we build an $(n+1)$-colored arena $G$ such that $\varphi$ is a tautology iff there exists a winning strategy for player 1 in $G$.

This result holds for all three goals we propose.

Assume $\varphi$ has $m$ variables.
Then, the arena $G$ is built as a cyclic concatenation of $m$ direct graphs with one entering node and one exiting node.

The entering and exiting nodes of each block are associated to player 0, while all other nodes to player 1.

# Outline

# Conclusion

In this work...

- we have studied two player games on colored graphs where the objective of player 1 is the construction of a balanced, bounded or, frequency-$f$ path,
- we have proved a sufficient condition for the existence of, at least, one player satisfying the memoryless strategy property,
- finally, we have constructed an algorithm that decides whether there exists a winning strategy for player 1, proving also that such a problem is CONPTIME-COMPLETE.

Thank you very much for your attention!
I hope my talk was not too monotone and selective.