

## REASONING ABOUT STRATEGIES: ON THE SATISFIABILITY PROBLEM\*

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**ABSTRACT.** *Strategy Logic* (SL, for short) has been introduced by Mogavero, Murano, and Vardi as a useful formalism for reasoning explicitly about strategies, as first-order objects, in multi-agent concurrent games. This logic turns out to be very powerful, subsuming all major previously studied modal logics for strategic reasoning, including ATL, ATL\*, and the like. Unfortunately, due to its high expressiveness, SL has a non-elementarily decidable model-checking problem and the satisfiability question is undecidable, specifically  $\Sigma_1^1$ -HARD.

In order to obtain a decidable sublogic, we introduce and study here *One-Goal Strategy Logic* (SL[1G], for short). This is a syntactic fragment of SL, strictly subsuming ATL\*, which encompasses formulas in prenex normal form having a single temporal goal at a time, for every strategy quantification of agents. We prove that, unlike SL, SL[1G] has the bounded tree-model property and its satisfiability problem is decidable in 2EXPTIME, thus not harder than the one for ATL\*.

### 1. INTRODUCTION

In open-system verification [CGP02, KVV01], an important area of research is the study of modal logics for strategic reasoning in the setting of multi-agent games [AHK02, JvdH04, Pau02, AGJ07, WvdHW07, BJ14, JM14]. An important contribution in this field has been the development of *Alternating-Time Temporal Logic* (ATL\*, for short), introduced by Alur, Henzinger, and Kupferman [AHK02]. ATL\* allows reasoning about strategic behavior of

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agents with temporal goals. Formally, it is obtained as a generalization of the branching-time temporal logic CTL<sup>\*</sup> [EH86], where the path quantifiers *there exists* “E” and *for all* “A” are replaced with strategic modalities of the form “ $\langle\langle A \rangle\rangle$ ” and “ $[\![A]\!]$ ”, for a set A of *agents*. Such strategic modalities are used to express cooperation and competition among agents in order to achieve certain temporal goals. In particular, these modalities express selective quantifications over those paths that are the results of infinite games between a coalition and its complement. ATL<sup>\*</sup> formulas are interpreted over *concurrent game structures* (CGS, for short) [AHK02], which model interacting processes. Given a CGS  $\mathcal{G}$  and a set A of agents, the ATL<sup>\*</sup> formula  $\langle\langle A \rangle\rangle\psi$  holds at a state  $s$  of  $\mathcal{G}$  if there is a set of strategies for the agents in A such that, no matter which strategies are executed by the agents not in A, the resulting outcome of the interaction in  $\mathcal{G}$  satisfies  $\psi$  at  $s$ . Several decision problems have been investigated about ATL<sup>\*</sup>; both its model-checking and satisfiability problems are decidable in 2EXPTIME [Sch08]. The complexity of the latter is just like the one for CTL<sup>\*</sup> [EJ88, EJ99].

Despite its powerful expressiveness, ATL<sup>\*</sup> suffers from the strong limitation that strategies are treated only implicitly through modalities that refer to games between competing coalitions. To overcome this problem, Chatterjee, Henzinger, and Piterman introduced *Strategy Logic* (CHP-SL, for short) [CHP07, CHP10], a logic that treats strategies in *two-player turn-based games* as *first-order objects*. The explicit treatment of strategies in this logic allows the expression of many properties not expressible in ATL<sup>\*</sup>. Although the model-checking problem of CHP-SL is known to be decidable with a non-elementary upper bound, it is not known whether the satisfiability problem is decidable as well [CHP10]. While the basic idea exploited in [CHP10] of explicitly quantifying over strategies is powerful and useful [FKL10], CHP-SL still suffers from various limitations. In particular, it is limited to two-player turn-based games. Furthermore, CHP-SL does not allow different players to share the same strategy, suggesting that strategies have yet to become truly first-class objects in this logic. For example, it is impossible to describe the classic strategy-stealing argument of combinatorial games such as Hex and the like.

These considerations led us to introduce and investigate a new *Strategy Logic*, denoted SL, as a more general framework than CHP-SL, for explicit reasoning about strategies in multi-agent concurrent games [MMV10a, MMPV14]. Syntactically, SL extends the linear-time temporal logic LTL [Pnu77] by means of *strategy quantifiers*, the existential  $\langle\langle x \rangle\rangle$  and the universal  $[\![x]\!]$ , as well as *agent binding*  $(a, x)$ , where  $a$  is an agent and  $x$  a variable. Intuitively, these elements can be read as “*there exists a strategy  $x$* ”, “*for all strategies  $x$* ”, and “*bind agent  $a$  to the strategy associated with  $x$* ”, respectively. For example, in a CGS  $\mathcal{G}$  with agents  $\alpha$ ,  $\beta$ , and  $\gamma$ , consider the property “ $\alpha$  and  $\beta$  have a common strategy to avoid a failure”. This property can be expressed by the SL formula  $\langle\langle \mathbf{x} \rangle\rangle [\![\mathbf{y}]\!](\alpha, \mathbf{x})(\beta, \mathbf{x})(\gamma, \mathbf{y})(\mathbf{G}\neg fail)$ . The variable  $\mathbf{x}$  is used to select a strategy for the agents  $\alpha$  and  $\beta$ , while  $\mathbf{y}$  is used to select another one for agent  $\gamma$  such that their composition, after the binding, results in a play where *fail* is never met. In [MMPV14] it has been showed that SL is very expressive and can represent several solution concepts. However, this high expressiveness comes at a price. Indeed, it has been shown in [MMPV14] that the model-checking problem is non-elementarily decidable. In particular, this problem is  $k$ -EXPSpace-HARD in the alternation number  $k$  of quantifications in the specification.

In this paper we investigate the satisfiability problem and some basic model-theoretic properties for SL. Regarding the former, as main result we show that SL is *highly undecidable*, precisely,  $\Sigma_1^1$ -HARD. Regarding the latter, we show that SL does not have the

bounded-tree model property.

The contrast between the undecidability of the satisfiability problem for SL and the elementary decidability of the same problem for  $\text{ATL}^*$ , provides motivation for an investigation of decidable fragments of SL that subsume  $\text{ATL}^*$ .

We introduce here the syntactic fragment *One-Goal Strategy Logic* (SL[1G], for short), which encompasses formulas in a special prenex normal form having a single temporal goal at a time. For goal we mean an SL formula of the type  $\mathfrak{b}\psi$ , where  $\mathfrak{b}$  is a binding prefix of the form  $(\alpha_1, x_1), \dots, (\alpha_n, x_n)$  containing all the involved agents and  $\psi$  is a formula in which every agent is not bounded to any variable, as for example an LTL specification. With SL[1G] one can express, for instance, visibility constraints on strategies among agents, i.e., only some agents from a coalition have knowledge of the strategies taken by those in the opponent coalition. Also, one can describe the fact that, in the Hex game, the strategy-stealing argument does not let the player who adopts it to win. Observe that the above properties cannot be expressed neither in  $\text{ATL}^*$  nor in CHP-SL.

In [MMPV14], we showed that SL[1G] is strictly more expressive than  $\text{ATL}^*$ , yet its model-checking problem is 2EXPTIME-COMplete, just like the one for  $\text{ATL}^*$ , while the same problem for SL is non-elementarily decidable. Our main result here is that the satisfiability problem for SL[1G] is also 2EXPTIME-COMplete. Thus, in spite of its expressiveness, SL[1G] has the same computational properties of  $\text{ATL}^*$ , which suggests that the one-goal restriction is the key to the elementary complexity of the latter logic too.

To achieve our main result, we use a fundamental property of the semantics of SL[1G] called *behavioral*<sup>1</sup>, which allows us to simplify reasoning about strategies by reducing it to a set of reasonings about actions. This intrinsic characteristic of SL[1G] means that, to choose an existentially quantified strategy, we do not need to know the entire structure of universally-quantified strategies, as it is the case for SL, but only their values on the histories of interest. Technically, to formally describe this property, we make use of the machinery of *dependence maps*, which is introduced to define a Skolemization procedure for SL, inspired by the one in first-order logic. By exploiting the behavioral property, one can show that SL[1G] satisfies the *bounded tree-model property*<sup>2</sup>. This allows us to efficiently make use of a *tree automata-theoretic approach* [Var96, VW86b] to solve the satisfiability problem. Given a formula  $\varphi$ , we build an *alternating co-Büchi tree automaton* [KVW00, MS95], whose size is only exponential in the size of  $\varphi$ , accepting all bounded-branching tree models of the formula with a suitable width. Then, together with the complexity of automata-nonemptiness checking, we get that the satisfiability procedure for SL[1G] is 2EXPTIME. We believe that our proof techniques are of independent interest and applicable to other logics as well.

**Related works.** Several works have focused on extensions of ATL and  $\text{ATL}^*$  to incorporate more powerful strategic constructs. Among them, we recall *Alternating-Time  $\mu$ CALCULUS* (AMuCalculus, for short) and *Game Logic* (GL, for short) [AHK02], *Quantified Decision Modality  $\mu$ CALCULUS* (QDMuCalculus, for short) [Pin07], *Coordination Logic* (CL, for short) [FS10], (*ATL with plausibility* (ATLP, for short) [BJD08], (*ATL with Irrevocable strategies* (IATL, for short) [AGJ07], (*Memoryful  $\text{ATL}^*$*  (mATL\*, for short) [MMV10b,

<sup>1</sup>We use this term as it has a direct correspondence with the “behavioral” concept used in game theory [Mye97, MMS13, MMS14].

<sup>2</sup>In [MMPV12], we indeed make use of a non-trivial proof to show this. In this paper, instead, we avoid the burden by making use of a recent result proved in [MP15].

MMV16], *Basic Strategy-Interaction Logic* (BSIL, for short) [WHY11] *Temporal Cooperation Logic* (TCL, for short) [HSW13], *Alternating-time Temporal Logic with Explicit Actions* (ATLEA, for short) [HLW13] and some extensions of  $\text{ATL}^*$  considered in [BLLM09]. AMu-Calculus and QDMuCalculus are intrinsically different from SL (as well as from CHP-SL and  $\text{ATL}^*$ ) as they are obtained by extending the propositional  $\mu$ -calculus [Koz83] with strategic modalities. CL is similar to QDMuCalculus but with LTL temporal operators instead of explicit fixpoint constructors. GL is strictly included in CHP-SL, in the case of two-player turn-based games, but it does not use any explicit treatment of strategies, as well as the extensions of  $\text{ATL}^*$  introduced in [BLLM09], which consider restrictions on the memory for strategy quantifiers. ATLP enables to express rationality assumptions of intelligent agents in ATL. In IATL, the semantics of the logic ATL is changed in a way that, in the evaluation of the goal, agents can be forced to keep the strategy they have chosen in the past in order to reach the state where a goal is evaluated.  $\text{mATL}^*$  enriches  $\text{ATL}^*$  by giving the ability to agents to “relent” and change their goals and strategies depending on the history of the play. BSIL allows to specify behaviors of a system that can cooperate with several strategies of the environment for different requirements. TCL extends ATL by allowing successive definitions of agent strategies, with the aim of using the collaborative power of groups of agents to enforce different temporal objectives. ATLEA introduces explicit actions in the logic ATL to reason about abilities of agents under commitments to play precise actions. Thus, all above logics are different from SL.

At roughly the time we have conceived Strategy Logic, another generalization of  $\text{ATL}^*$ , named  $\text{ATL}^*$  *with Strategy Contexts*, which turns out to be very expressive but a proper sublogic of SL, has been considered in [DLM10] (see also [DLM12, TW12, LM13, LM15] for more recent works). In this logic, a quantification over strategies does not reset the strategies previously quantified but allows to maintain them in a particular context in order to be reused. This makes the logic much more expressive than  $\text{ATL}^*$ .

Recently, several extensions of SL have been also investigated. *Updatable Strategy Logic* (USL, for short) has been considered in [CBC13, CBC15] where, in addition to SL, an agent can refine its own strategies by means of an “unbinder” operator, which explicitly deletes the binding of a strategy to an agent. In [Bel14, ČLMM14], an epistemic extension of SL with modal operators for individual knowledge has been considered, showing that the complexity of model checking for this logic is not worse than the one for (non-epistemic) SL. Last but not least, in [CLM15] a BDD-based model checker for the verification of systems against specifications expressed in  $\text{SL}[1G]$  has been introduced (see also [ČLMM14] for an introduction to the conceived tool).

Finally, worth of mention are the works handling the synthesis question of specifications expressed in the logic CHP-SL, as well as logics related to SL. Among the others, we report the works [CDFR14, FKL10, KPV14, GHW14, KPV16].

Outline. In Section 2, we first introduce the syntax of SL, as well as the notion of *Concurrent Game Structure*, on which the logic is interpreted. We also provide examples to show useful applications of the logic in the context of formal verification. In Section 3 we show that the satisfiability problem for SL is highly undecidable. We do this by first proving that the logic does not have the bounded-model property and then providing a reduction of the satisfiability problem from the *recurrent domino problem*, which has been proved to be undecidable by Harel in [Har84]. Given this negative result, in Section 4, we investigate on the theoretical properties that make  $\text{ATL}^*$  decidable. We first recall the definition of two

syntactic fragments of SL, which we call *Boolean-Goal Strategy Logic* (SL[BG]) and *One-Goal Strategy Logic* (SL[1G]), showing that the former retains all the negative properties of SL, while the latter satisfies a fundamental property, namely the *behavioral* semantics, that turns out to be fundamental, in Section 5, to prove that SL[1G] enjoys the bounded-model property and a decidable satisfiability problem. In order to prove the last result, we employ an automata-theoretic approach, from which we derive a 2EXPTIME procedure.

## 2. STRATEGY LOGIC

*Strategy Logic* [MMV10a] (SL, for short) is an extension of the classic linear-time temporal logic LTL [Pnu77] along with the concepts of strategy quantifications and agent binding, which formalism allows to express strategic plans over temporal goals. The main distinctive feature of this formalism *w.r.t.* other logics with the same aim resides in the decoupling strategy instantiations, done through the quantifications, from their applications, by means of bindings. Consequently, the logic is not simply propositional but predicative, since we treat strategies as a first order concept via the use of agents and variables as explicit syntactic elements. This fact allows us to write Boolean combinations and nesting of complex predicates, each one representing a different temporal goal, linked together by some common strategic choices.

The section is organized as follows. In Subsection 2.1, we recall the definition of concurrent game structure used to interpret SL, whose syntax is reported in Subsection 2.2. Then, in Subsection 2.3, we give, among the others, the notions of strategy, assignment, and play, which are finally used to define the semantics of the logic in Subsection 2.4.

**2.1. Underlying framework.** As semantic framework for SL, we use the *graph-based model* for *multi-player games* named *concurrent game structure* [AHK02], which is a generalization of *Kripke structures* [Kri63] and *labeled transition systems* [Kel76]. It allows to model *multi-agent systems* viewed as extensive form games, in which players perform *concurrent actions* to trigger different transitions over the graph.

**Definition 2.1** (Concurrent Game Structures). A *concurrent game structure* (CGS, for short) is a tuple  $\mathcal{G} \triangleq \langle \text{AP}, \text{Ag}, \text{Ac}, \text{St}, \text{tr}, \text{ap}, s_o \rangle$ , where AP and Ag are finite non-empty sets of *atomic propositions* and *agents*, Ac and St are enumerable non-empty sets of *actions* and *states*,  $s_o \in \text{St}$  is a designated *initial state*, and  $\text{ap} : \text{St} \rightarrow 2^{\text{AP}}$  is a *labeling function* that maps each state to the set of atomic propositions true in that state. Let  $\text{Dc} \triangleq \text{Ac}^{\text{Ag}}$  be the set of *decisions*, *a.k.a.* *action profiles* in the literature, *i.e.*, functions from Ag to Ac representing the choices of an action for each agent.<sup>3</sup> Then,  $\text{tr} : \text{St} \times \text{Dc} \rightarrow \text{St}$  is a *transition function* mapping a pair of a state and a decision to a state.

**Remark 2.2.** The reader might note that the definition of CGS given here differs to the one provided in the literature, for example in [AHK02], from the fact that set of actions is not specified for each agent. On one hand, this has the advantage of simplifying the notation and the technical development of the results. On the other hand, this looks less general from the mechanism design point of view, in which the roles of the agents are not always symmetric, and it turns to be useful to assign a specific set of action per each agent.

<sup>3</sup>In the following, we use both  $X \rightarrow Y$  and  $Y^X$  to denote the set of functions from the domain X to the codomain Y.

However, by means of a suitable mapping of actions, it is not hard to show that the general case of CGS described in [AHK02] can be accounted in Definition 2.1.

To get familiar with the concept of CGS, we present here some running example of simple concurrent games.

First, we analyze an extended version of the well-known prisoner’s dilemma [OR94] in which also the actions of the police are taken into account.

**Example 2.3** (Prisoners and Police’s Dilemma).

In the *prisoner’s dilemma* (PD, for short), two accomplices are interrogated in separated rooms by the police, which offers them the same agreement. If one defects, i.e., testifies for the prosecution against the other, while the other cooperates, i.e., remains silent, the defector goes free and the silent accomplice goes to jail. If both cooperate, they remain free, but will be surely interrogated in the next future waiting for a defection. On the other hand, if they both defect, both go to jail. In the *prisoner and police’s dilemma* (PPD, for short), apart from the classic agreement of the PD, the two accomplices also know that they can try to gain a better sentence by the judge if one spontaneously defects without being interrogated by police, since he is considered a “good willing man”. In this case, indeed, if the other cooperates, the defector becomes definitely free, while the other goes to jail with the possibility to eventually be released. It is important, however, that neither of them defects, otherwise the police can subtly act as they were interrogated. Moreover, differently from the PD, they are not free during the time in which they can be interrogated. This complex situation, can be modeled by the CGS  $\mathcal{G}_{PPD} \triangleq \langle AP, Ag, Ac, St, tr, ap, s_0 \rangle$  depicted in Figure 1, where there are three agents in  $Ag \triangleq \{A_1, A_2, P\}$ , with P being the police, and all of them can execute the two abstract actions in  $Ac \triangleq \{0, 1\}$ . For the accomplices, 0 and 1 have the meaning of “cooperate” and “defect”. For the police, on the contrary, they mean “wait” and “interrogate” in all the states but those in which one of the accomplices can eventually be released, where the meaning is “release” and “maintain”, instead. The set of states for the game is given by  $\{s_i, s_{A_1}, s_{A_2}, s_j, s_{A_1j}, s_{A_2j}, s_{A_1A_2}\}$ . The idle state  $s_i$  denotes the situation in which the two prisoners are waiting to be interrogated by police. They can even decide to defect before interrogation. The states  $s_{A_1}$  and  $s_{A_2}$  denotes the situation in which only one prisoner becomes definitely free. Moreover, the states  $s_{A_1j}$  and  $s_{A_2j}$  indicate when one of the prisoners is free while the other in the jail is waiting for his release. Finally,  $s_{A_1A_2}$  denotes the state in which both prisoners have gained definitely the freedom. To represent the different meaning of these states, we use the atomic propositions  $f_{A_i}$  to denote that the prisoner  $A_i$  is free. Both the labeling function  $ap$  and the transition function  $tr$  can be extracted from the figure,

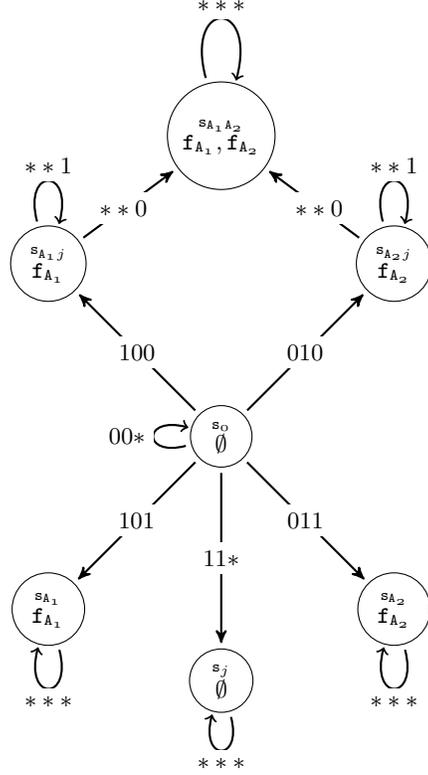


Figure 1: The CGS  $\mathcal{G}_{PPD}$ .

where the agents  $A_1$ ,  $A_2$ , and  $P$  control the first, second and third components of the triple actions over the edges, respectively.

In addition to PPD, we model a very simple *preemptive scheduling* protocol for the access of processes to a shared resource.

**Example 2.4** (Preemptive Scheduling). Consider the following *preemptive scheduling* protocol (PS, for short) describing the access rules of two processes to a shared resource in a preemptive way. When the resource is free and only one process asks for it, this process directly receives the grant. Instead, if there is a competition of requests, it is the scheduler that, in a nondeterministic way, determines who can access to the resource. Finally, in case one process owns the resource while the other asks for it, the scheduler can choose whether to apply a preemption. These rules are formalized in the CGS  $\mathcal{G}_{PS} \triangleq \langle AP, Ag, Ac, St, tr, ap, s_0 \rangle$  of Figure 2, where the agents “Process-1”, “Process-2” and “Scheduler” in  $Ag \triangleq \{P_1, P_2, S\}$  can choose between the two abstract actions in  $Ac \triangleq \{0, 1\}$ . The processes use the actions 0 to not send any request and 1 to send a request to the scheduler, while the scheduler uses them in order to decide who can have the access to the resource in a situation of competition. There are five states  $St \triangleq \{s_i, s_1, s_2, s_{1,2}, s'_1, s'_2\}$  in which the protocol can reside: the idle state  $s_i$  in which the resource is free; the three states  $s_1$ ,  $s_2$ , and  $s_{1,2}$  in which  $P_1$ ,  $P_2$  or both are requesting the resource; the two states  $s'_1$  and  $s'_2$  in which the resource has been finally granted to  $P_1$  and  $P_2$ , respectively. To represent all information associated, we use the atomic propositions in  $AP \triangleq \{r_1, r_2, g_1, g_2\}$ , where  $r_i$  represents the request of  $P_i$ , while  $g_i$  the fact that the resource has been granted to  $P_i$ .

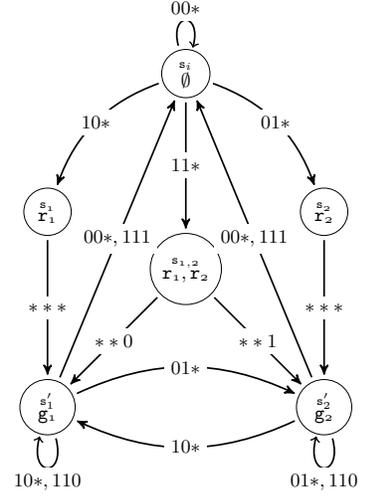


Figure 2: The CGS  $\mathcal{G}_{PS}$ .

**2.2. Syntax.** *Strategy Logic* (SL, for short) syntactically extends LTL by means of two *strategy quantifiers*, the existential  $\langle\langle x \rangle\rangle$  and the universal  $\llbracket x \rrbracket$ , and the *agent binding*  $(a, x)$ , where  $a$  is an agent and  $x$  a variable. Intuitively, these new elements can be read as “there exists a strategy  $x$ ”, “for all strategies  $x$ ”, and “bind agent  $a$  to the strategy associated with the variable  $x$ ”, respectively. The formal syntax of SL follows.

**Definition 2.5** (SL Syntax). SL *formulas* are built inductively from the sets of atomic propositions  $AP$ , variables  $Vr$ , and agents  $Ag$ , by using the following grammar, where  $p \in AP$ ,  $x \in Vr$ , and  $a \in Ag$ :

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid X\varphi \mid \varphi U\varphi \mid \varphi R\varphi \mid \langle\langle x \rangle\rangle\varphi \mid \llbracket x \rrbracket\varphi \mid (a, x)\varphi.$$

SL denotes the infinite set of formulas generated by the above rules.

Observe that, by construction, LTL is a proper syntactic fragment of SL, *i.e.*,  $LTL \subset SL$ . In order to abbreviate the writing of formulas, we use the boolean values true  $\mathbf{t}$  and false  $\mathbf{f}$  and the well-known temporal operators future  $F\varphi \triangleq \mathbf{t} U\varphi$  and globally  $G\varphi \triangleq \mathbf{f} R\varphi$ . Moreover, we use the italic letters  $x, y, z, \dots$ , possibly with indexes, as meta-variables on the variables  $\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$  in  $Vr$ .

A first classic notion related to the syntax of SL is that of *subformula*, *i.e.*, a syntactic expression that is part of an a priori given formula. By  $\text{sub}(\varphi)$  we formally denote the set of subformulas of an SL formula  $\varphi$ . For instance, consider  $\varphi = \langle\langle \mathbf{x} \rangle\rangle(\alpha, \mathbf{x})(\mathbf{Fp})$ . Then, it is immediate to see that  $\text{sub}(\varphi) = \{\varphi, (\alpha, \mathbf{x})(\mathbf{Fp}), (\mathbf{Fp}), \mathbf{p}, \mathbf{t}\}$ .

Usually, in predicative logics, we need the concepts of *free* and *bound* placeholders, to correctly define the meaning of a formula. In SL, we have two different kind of placeholders: variables and agents. The former is used in the strategy quantifications, the latter to commit an agent, by means of bindings, to adhere to a strategy. Consequently, we need to differentiate the sets of free variables and free agents of an SL formula  $\varphi$ . The first contains the variables that are not in a scope of a quantification. The second, instead, contains the agents for which there is no related binding in the scope of a temporal operator. A formula without any free variable (*resp.*, agent) is named *variable-closed* (*resp.*, *agent-closed*). A formula that is both variable- and agent-closed, is named *sentence*. For a given SL formula  $\varphi$ , by  $\text{free}(\varphi)$  we denote the set of both free variables and agents occurring in  $\varphi$ . The formal definition of  $\text{free}(\cdot)$ , which we report in the following, has been given in [MMPV14].

**Definition 2.6** (SL Free Agents/Variables). The set of *free agents/variables* of an SL formula is given by the function  $\text{free} : \text{SL} \rightarrow 2^{\text{Ag} \cup \text{Vr}}$  defined as follows:

- (i)  $\text{free}(p) \triangleq \emptyset$ , where  $p \in \text{AP}$ ;
- (ii)  $\text{free}(\neg\varphi) \triangleq \text{free}(\varphi)$ ;
- (iii)  $\text{free}(\varphi_1 \text{Op} \varphi_2) \triangleq \text{free}(\varphi_1) \cup \text{free}(\varphi_2)$ , where  $\text{Op} \in \{\wedge, \vee\}$ ;
- (iv)  $\text{free}(X\varphi) \triangleq \text{Ag} \cup \text{free}(\varphi)$ ;
- (v)  $\text{free}(\varphi_1 \text{Op} \varphi_2) \triangleq \text{Ag} \cup \text{free}(\varphi_1) \cup \text{free}(\varphi_2)$ , where  $\text{Op} \in \{\text{UR}\}$ ;
- (vi)  $\text{free}(\text{Qn}\varphi) \triangleq \text{free}(\varphi) \setminus \{x\}$ , where  $\text{Qn} \in \{\langle\langle x \rangle\rangle, \llbracket x \rrbracket : x \in \text{Vr}\}$ ;
- (vii)  $\text{free}((a, x)\varphi) \triangleq \text{free}(\varphi)$ , if  $a \notin \text{free}(\varphi)$ , where  $a \in \text{Ag}$  and  $x \in \text{Vr}$ ;
- (viii)  $\text{free}((a, x)\varphi) \triangleq (\text{free}(\varphi) \setminus \{a\}) \cup \{x\}$ , if  $a \in \text{free}(\varphi)$ , where  $a \in \text{Ag}$  and  $x \in \text{Vr}$ .

Observe that, on one hand, free agents are introduced in Items ((iv)) and ((v)) and removed in Item ((viii)). On the other hand, free variables are introduced in Item ((viii)) and removed in Item ((vi)).

As an example, let  $\varphi = \langle\langle \mathbf{x} \rangle\rangle(\alpha, \mathbf{x})(\beta, \mathbf{y})(\mathbf{Fp})$  be a formula on the agents  $\text{Ag} = \{\alpha, \beta, \gamma\}$ . Then, we have  $\text{free}(\varphi) = \{\gamma, \mathbf{y}\}$ , since  $\gamma$  is an agent without any binding after  $\mathbf{Fp}$  and  $\mathbf{y}$  has no quantification at all. Consider also the formulas  $(\alpha, \mathbf{z})\varphi$  and  $(\gamma, \mathbf{z})\varphi$ , where the subformula  $\varphi$  is the same as above. Then, we have  $\text{free}((\alpha, \mathbf{z})\varphi) = \{\gamma, \mathbf{y}, \mathbf{z}\}$  and  $\text{free}((\gamma, \mathbf{z})\varphi) = \{\mathbf{y}, \mathbf{z}\}$ , since  $\alpha$  is not free in  $\varphi$  but  $\gamma$  is, *i.e.*,  $\alpha \notin \text{free}(\varphi)$  and  $\gamma \in \text{free}(\varphi)$ . So,  $(\gamma, \mathbf{z})\varphi$  is agent-closed while  $(\alpha, \mathbf{z})\varphi$  is not.

In order to practice with the syntax of SL, we now describe few examples of some game-theoretic properties, which cannot be expressed neither in  $\text{ATL}^*$  nor in  $\text{CHP-SL}$ . We clarify this point later in the paper. The interpretation of these formulas is quite intuitive. Leastwise, the reader can rely on the formal semantics, which is given later in the paper.

The first we introduce is the well-known concept of *Nash Equilibrium* in concurrent infinite games with Boolean payoffs.

**Example 2.7** (Nash Equilibrium). Consider the  $n$  agents  $\alpha_1, \dots, \alpha_n$  of a game, each of them having, respectively, a possibly different temporal goal described by one of the LTL formulas  $\psi_1, \dots, \psi_n$ . Then, we can express the existence of a strategy profile  $(\mathbf{x}_1, \dots, \mathbf{x}_n)$

that is a *Nash equilibrium* (NE, for short) for  $\alpha_1, \dots, \alpha_n$  *w.r.t.*  $\psi_1, \dots, \psi_n$  by using the SL sentence  $\varphi_{NE} \triangleq \langle\langle \mathbf{x}_1 \rangle\rangle \cdots \langle\langle \mathbf{x}_n \rangle\rangle (\alpha_1, \mathbf{x}_1) \cdots (\alpha_n, \mathbf{x}_n) \psi_{NE}$ , where  $\psi_{NE} \triangleq \bigwedge_{i=1}^n (\langle\langle \mathbf{y} \rangle\rangle (\alpha_i, \mathbf{y}) \psi_i) \rightarrow \psi_i$  is a variable-closed formula. Informally, this asserts that every agent  $\alpha_i$  has  $\mathbf{x}_i$  as one of the best strategy *w.r.t.* the goal  $\psi_i$ , once all the other strategies of the remaining agents  $\alpha_j$ , with  $j \neq i$ , have been fixed to  $\mathbf{x}_j$ . Note that here we are only considering equilibria under deterministic strategies.

In a game in which not all agents are peers, we can have one or more of them that may vary the payoff of the others, without having a personal aim, *i.e.*, without looking for the maximization of their own payoffs. Such situations can usually arise when we have games with arbiters or similar characters, like supervisors or government authorities, that have to be fair, *i.e.*, they have to lay down an *equity governance*.

**Example 2.8** (Equity Governance). Consider a game similar to the one described in the previous example, in which there is, in addition, a supervisor agent  $\beta$  that does not have a specific goal. However, the peers want him to be fair *w.r.t.* their own goals, *i.e.*, the supervisor has to use a strategy that must not prefer one agent over another. This concept is called *equity governance* (EG, for short). In order to formalize it, we can use the SL sentence  $\varphi_{EG} \triangleq \llbracket \mathbf{x}_1 \rrbracket \cdots \llbracket \mathbf{x}_n \rrbracket (\alpha_1, \mathbf{x}_1) \cdots (\alpha_n, \mathbf{x}_n) \langle\langle \mathbf{y} \rangle\rangle (\beta, \mathbf{y}) \psi_{EG}$ , where  $\psi_{EG} \triangleq \bigwedge_{i,j=1, i < j}^n (\langle\langle \mathbf{z}_1 \rangle\rangle (\beta, \mathbf{z}_1) \psi_i) \wedge (\langle\langle \mathbf{z}_2 \rangle\rangle (\beta, \mathbf{z}_2) \psi_j) \rightarrow (\psi_i \leftrightarrow \psi_j)$ . Informally, the  $\psi_{EG}$  subformula asserts that, if there are two strategies  $\mathbf{z}_1$  and  $\mathbf{z}_2$  for  $\beta$  that allow  $\alpha_i$  and  $\alpha_j$  to achieve their own goals  $\psi_i$  and  $\psi_j$ , separately, then the unique strategy  $\mathbf{y}$  previously chosen by the supervisor has to ensure the achievement of either both the goals or none of them. Note that the sentence  $\varphi_{EG}$  requires the existence of an EG strategy  $\mathbf{y}$  for  $\beta$ , in dependence of the strategies  $\mathbf{x}_1, \dots, \mathbf{x}_n$  chosen by the peers. To verify the existence of a uniform EG, we may use the SL sentence  $\varphi_{UEG} \triangleq \langle\langle \mathbf{y} \rangle\rangle (\beta, \mathbf{y}) \psi'_{EG}$ , with  $\psi'_{EG} \triangleq \llbracket \mathbf{x}_1 \rrbracket \cdots \llbracket \mathbf{x}_n \rrbracket (\alpha_1, \mathbf{x}_1) \cdots (\alpha_n, \mathbf{x}_n) \psi_{EG}$ , whose difference *w.r.t.*  $\varphi_{EG}$  resides only in the alternation of quantifiers. Finally, to verify the existence of a uniform EG that allows also the existence of an NE for the peers, we can use the SL sentence  $\varphi_{UEG+NE} \triangleq \langle\langle \mathbf{y} \rangle\rangle (\beta, \mathbf{y}) (\psi'_{EG} \wedge \varphi_{NE})$ .

Usually, the fairness of a supervisor does not ensure that the whole game can advance, *i.e.*, that the peers can achieve their respective goals. Indeed, there are games like the zero-sum ones in which the agents have opposite goals that cannot be achieved at the same time. However, there are different kind of games, as the PPD or the PS of Examples 2.3 and 2.4, in which a supervisor can try to help all peers in their intent, by applying an *advancement governance*.

**Example 2.9** (Advancement Governance). Consider the game described in the previous example of EG. Here, we want to consider an *advancement governance* (AG, for short) for the supervisor, *i.e.*, a strategy for  $\beta$  that allows the peers to achieve their own goals, if they have the will and possibility to do so. Formally, this concept can be expressed by using the SL sentence  $\varphi_{AG} \triangleq \llbracket \mathbf{x}_1 \rrbracket \cdots \llbracket \mathbf{x}_n \rrbracket (\alpha_1, \mathbf{x}_1) \cdots (\alpha_n, \mathbf{x}_n) \langle\langle \mathbf{y} \rangle\rangle (\beta, \mathbf{y}) \psi_{AG}$ , where  $\psi_{AG} \triangleq (\bigwedge_{i=1}^n (\langle\langle \mathbf{z} \rangle\rangle (\beta, \mathbf{z}) \psi_i) \rightarrow \psi_i)$ . Intuitively, the  $\psi_{AG}$  subformula expresses the fact that, if  $\beta$  has a strategy  $\mathbf{z}$  able to force a goal  $\psi_i$ , once the strategies of the peers  $\alpha_1, \dots, \alpha_n$  have been fixed, then his a priori choice  $\mathbf{y}$  *w.r.t.* the goals has to force  $\psi_i$  as well. As in the case of EG, we can have an uniform version of AG, by using the SL sentence  $\varphi_{UAG} \triangleq \langle\langle \mathbf{y} \rangle\rangle (\beta, \mathbf{y}) \psi'_{AG}$ , where  $\psi'_{AG} \triangleq \llbracket \mathbf{x}_1 \rrbracket \cdots \llbracket \mathbf{x}_n \rrbracket (\alpha_1, \mathbf{x}_1) \cdots (\alpha_n, \mathbf{x}_n) \psi_{AG}$ .

Differently from the previous examples, one can consider the case in which the authority agent has his own goal to be satisfied, provided the other agents to be in a certain equilibrium.

In the context of system design [PR89], *rational synthesis* [FKL10, KPV14, KPV16] is a recent improvement of the classical reactive one. In this setting, the adversarial environment is not a monolithic block, but a set of agent components, each of them having their own goal. In the next example, we show that the most typical instances of a rational synthesis problem can be represented in SL.

**Example 2.10** (Rational Synthesis). Consider a solution concept that is representable in SL by means of a suitable formula  $\psi_{SC}$ , *e.g.*, NE, and a temporal goal  $\psi_\beta$  for the system agent. Here, we look for a *rational synthesis* solution for the players, *i.e.*, a strategy profile  $(y, x_1, \dots, x_n)$  such that, if  $\beta$  acts according to  $y$ , then  $\psi_\beta$  is satisfied and  $(x_1, \dots, x_n)$  is in an equilibrium according to the solution concept considered, *i.e.*,  $\psi_{SC}$  is satisfied. Formally, this concept can be expressed by using the SL sentence  $\varphi_{RS} = \langle\langle y \rangle\rangle \langle\langle x_1 \rangle\rangle \dots \langle\langle x_n \rangle\rangle (\beta, y)(a_1, x_1) \dots (a_n, x_n)(\psi_0 \wedge \psi_{SC})$ . As an example,  $\psi_{SC}$  can be the formula  $\psi_{NE}$  of Example 2.7. In this case, we obtain the rational synthesis problem for NE.

**2.3. Basic notions.** Before continuing with the formal description of SL, we need to introduce some basic notions related to CGSs, such as those of *track*, *path*, *strategy*, and the like. All these notions have been already introduced in [MMV10b]. However, for the sake of completeness, as well as for their importance in the definition of SL semantics, we fully report them in this section.

We start with the notions of *track* and *path*. Intuitively, tracks and paths of a CGS are legal sequences of reachable states that can be respectively seen as partial and complete descriptions of possible outcomes of the game modeled by the structure itself. Formally, a *track* (resp., *path*) in a CGS  $\mathcal{G}$  is a finite (resp., an infinite) sequence of states  $\rho \in \text{St}^*$  (resp.,  $\pi \in \text{St}^\omega$ ) such that, for all  $i \in [0, |\rho| - 1[$  (resp.,  $i \in \mathbb{N}$ ), there exists a decision  $\delta \in \text{Dc}$  such that  $(\rho)_{i+1} = \text{tr}((\rho)_i, \delta)$  (resp.,  $(\pi)_{i+1} = \text{tr}((\pi)_i, \delta)$ )<sup>4</sup>. A track  $\rho$  is *non-trivial* if it has non-zero length, *i.e.*,  $|\rho| > 0$  that is  $\rho \neq \epsilon$ <sup>5</sup>. The set  $\text{Trk} \subseteq \text{St}^+$  (resp.,  $\text{Pth} \subseteq \text{St}^\omega$ ) contains all non-trivial tracks (resp., paths). Moreover,  $\text{Trk}(s) \triangleq \{\rho \in \text{Trk} : \text{fst}(\rho) = s\}$  (resp.,  $\text{Pth}(s) \triangleq \{\pi \in \text{Pth} : \text{fst}(\pi) = s\}$ ) indicates the subsets of tracks (resp., paths) starting at a state  $s \in \text{St}$ <sup>6</sup>. In some cases, to avoid any ambiguity, we use subscripts like  $\text{Trk}_{\mathcal{G}}$ ,  $\text{Pth}_{\mathcal{G}}$ , and so on, to denote the fact that we are referring to the set of tracks, paths, and the like, in  $\mathcal{G}$ .

As an example, consider the CGS  $\mathcal{G}_{PS}$  in Figure 2. Then  $\rho = s_i \cdot s_1 \cdot s_1' \cdot s_1' \cdot s_i$  and  $\pi = (s_i \cdot s_{1,2} \cdot s_1' \cdot s_i \cdot s_{1,2} \cdot s_2')^\omega$  are a track and a path, respectively. Moreover, we have that  $\text{Trk}_{\mathcal{G}_{PS}} = \text{St}_{\mathcal{G}_{PS}}^*$  and  $\text{Pth}_{\mathcal{G}_{PS}} = \text{St}_{\mathcal{G}_{PS}}^\omega$ .

At this point, we can define the concept of *strategy*. Intuitively, a strategy is a scheme for an agent that contains all choices of actions as in dependence of the history of the current outcome. However, observe that here there is not an a priori connection between a strategy and an agent, since the same strategy can be used by more than one agent at the same time. Formally, a *strategy* in a CGS  $\mathcal{G}$  is a function  $f : \text{Trk} \rightarrow \text{Ac}$  that maps each non-trivial track to an action. The set  $\text{Str} \triangleq \text{Trk} \rightarrow \text{Ac}$  contains all strategies.

An example of strategy in the CGS  $\mathcal{G}_{PS}$  is given by the function  $f_1 \in \text{Str}$  assigning the action 0 to all the tracks in which the state  $s_i$  occurs an odd number of times and the action

<sup>4</sup>The notation  $(w)_i \in \Sigma$  indicates the *element* of index  $i \in [0, |w|[$  of a non-empty sequence  $w \in \Sigma^\infty$ , where  $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ .

<sup>5</sup>The Greek letter  $\epsilon$  stands for the *empty sequence*.

<sup>6</sup>By  $\text{fst}(w) \triangleq (w)_0$  we denote the *first element* of an infinite sequence  $w \in \Sigma^\infty$ .

1, otherwise. Another example of strategy is the function  $f_2 \in \text{Str}$  assigning the action 1 on all possible tracks of the CGS.

We now introduce the notion of *assignment*. Intuitively, an assignment gives a valuation of variables with strategies, where the latter are used to determine the behavior of agents in the game. With more detail, as in the case of first order logic, we use this concept as a technical tool to quantify over strategies associated with variables, independently of agents to which they are related to. So, assignments are used precisely as a way to define a correspondence between variables and agents via strategies.

**Definition 2.11** (Assignments). An *assignment* in a CGS  $\mathcal{G}$  is a partial function  $\chi : \text{Vr} \cup \text{Ag} \rightarrow \text{Str}$  mapping variables and agents in its domain to a strategy. An assignment  $\chi$  is *complete* if it is defined on all agents, *i.e.*,  $\text{Ag} \subseteq \text{dom}(\chi)$ . The set  $\text{Asg} \triangleq \text{Vr} \cup \text{Ag} \rightarrow \text{Str}$  contains all assignments. Moreover,  $\text{Asg}(\text{X}) \triangleq \text{X} \rightarrow \text{Str}$  indicates the subset of *X-defined* assignments, *i.e.*, assignments defined on the set  $\text{X} \subseteq \text{Vr} \cup \text{Ag}$ .

As an example of assignment, consider the CGS  $\mathcal{G}_{PS}$  of Example 2.4 in which  $\text{Ag} = \{\text{P}_1, \text{P}_2\}$  and the function  $\chi_1 \in \text{Asg}$  in , with  $\text{dom}(\chi_1) = \{\text{P}_1, \mathbf{x}\}$ , such that  $\chi_1(\text{P}_1) = f_1$  and  $\chi_1(\mathbf{x}) = f_2$ . As another example, consider the assignment  $\chi_2 \in \text{Asg}$  in the same CGS, with  $\text{dom}(\chi_2) = \text{Ag}$ , such that  $\chi_2(\text{S}) = f_1$  and  $\chi_2(\text{P}_1) = \chi_2(\text{P}_2) = f_2$ . Note that  $\chi_2$  is complete, while  $\chi_1$  is not.

Given an assignment  $\chi \in \text{Asg}$ , an agent or variable  $l \in \text{Vr} \cup \text{Ag}$ , and a strategy  $f \in \text{Str}$ , we need to describe the *redefinition* of  $\chi$ , *i.e.*, a new assignment equal to the first one on all elements of its domain but  $l$ , on which it assumes the value  $f$ . Formally, with  $\chi[l \mapsto f] \in \text{Asg}$  we denote the new assignment defined on  $\text{dom}(\chi[l \mapsto f]) \triangleq \text{dom}(\chi) \cup \{l\}$  that returns  $f$  on  $l$  and is equal to  $\chi$  on the remaining part of its domain, *i.e.*,  $\chi[l \mapsto f](l) \triangleq f$  and  $\chi[l \mapsto f](l') \triangleq \chi(l')$ , for all  $l' \in \text{dom}(\chi) \setminus \{l\}$ . Intuitively, if we have to add or update a strategy that needs to be bound by an agent or variable, we can simply take the old assignment and redefine it by using the above notation.

Now, we can formalize the concept of *play* in a game. Intuitively, a play is the unique outcome of the game determined by all agent strategies participating to it.

**Definition 2.12** (Plays). A path  $\pi \in \text{Pth}(s)$  starting at a state  $s \in \text{St}$  is a *play w.r.t.* a complete assignment  $\chi \in \text{Asg}(s)$  ( $(\chi, s)$ -*play*, for short) if, for all  $i \in \mathbb{N}$ , it holds that  $(\pi)_{i+1} = \text{tr}((\pi)_i, \delta)$ , where  $\delta(a) \triangleq \chi(a)((\pi)_{\leq i})$ , for each  $a \in \text{Ag}$ <sup>7</sup>. The partial function  $\text{play} : \text{Asg} \times \text{St} \rightarrow \text{Pth}$ , with  $\text{dom}(\text{play}) \triangleq \{(\chi, s) : \text{Ag} \subseteq \text{dom}(\chi) \wedge \chi \in \text{Asg}(s) \wedge s \in \text{St}\}$ , returns the  $(\chi, s)$ -play  $\text{play}(\chi, s) \in \text{Pth}(s)$ , for all pairs  $(\chi, s)$  in its domain.

As last example, consider again the CGS  $\mathcal{G}_{PS}$  and the complete assignment  $\chi_2$  defined above. Then, we have that  $\text{play}(\chi_2, s_0) = (s_i \cdot s_{1,2} \cdot s_1' \cdot s_i \cdot s_{1,2} \cdot s_2')^\omega$ .

Finally, we give the definition of global translation of a complete assignment associated with a state, which is used to capture, at a certain step of the play, what is the current state and its updated assignment.

**Definition 2.13** (Global Translation). For a given state  $s \in \text{St}$  and a complete assignment  $\chi \in \text{Asg}$ , an *i-th global translation* of  $(\chi, s)$ , with  $i \in \mathbb{N}$ , is a pair of a complete assignment and a state  $(\chi, s)^i \triangleq ((\chi)_{(\pi)_{\leq i}}, (\pi)_i)$ , where  $\pi = \text{play}(\chi, s)$  and  $(\chi)_{(\pi)_{\leq i}}$  denotes an assignment such that, for all  $l \in \text{dom}(\chi)$ ,  $(\chi)_{(\pi)_{\leq i}}(l)(\rho) = \chi(l)((\pi)_{\leq i} \cdot \rho)$ , for all  $\rho \in \text{dom}((\chi)_{(\pi)_{\leq i}}(l))$ .

<sup>7</sup>The notation  $(w)_{\leq i} \in \Sigma^*$  indicates the *prefix* up to index  $i \in [0, |w|]$  of a non-empty sequence  $w \in \Sigma^\infty$ .

Intuitively, an  $i$ -th global translation of  $(s, \chi)$  is meant to return a pair of a state and a complete assignment  $(\chi, s)^i$  for which the play  $\text{play}((\chi, s)^i)$  generated corresponds to  $\text{play}((\chi, s))_{\geq i}$ , *i.e.*, the suffix from the  $i$ -th element of the play  $\text{play}(s, \chi)$ . This property will be used below to correctly define the semantics of the temporal operators in SL.

**2.4. Semantics.** As already reported at the beginning of this section, just like ATL<sup>\*</sup> and differently from CHP-SL, the semantics of SL is defined *w.r.t.* concurrent game structures. For an SL formula  $\varphi$ , a CGS  $\mathcal{G}$ , a state  $s$  in it, and an assignment  $\chi$  with  $\text{free}(\varphi) \subseteq \text{dom}(\chi)$ , we write  $\mathcal{G}, \chi, s \models \varphi$  to indicate that the formula  $\varphi$  holds at  $s$  in  $\mathcal{G}$  under  $\chi$ . The semantics of SL formulas involving the atomic propositions, the Boolean connectives  $\neg$ ,  $\wedge$ , and  $\vee$ , as well as the temporal operators **X**, **U**, and **R** is defined as usual in LTL. The novel part resides in the formalization of the meaning of strategy quantifications  $\langle\langle x \rangle\rangle$  and  $\llbracket x \rrbracket$  and agent binding  $(a, x)$ .

**Definition 2.14** (SL Semantics). Given a CGS  $\mathcal{G}$ , for all SL formulas  $\varphi$ , states  $s \in \text{St}$ , and assignments  $\chi \in \text{Asg}$  with  $\text{free}(\varphi) \subseteq \text{dom}(\chi)$ , the modeling relation  $\mathcal{G}, \chi, s \models \varphi$  is inductively defined as follows.

- (1)  $\mathcal{G}, \chi, s \models p$  if  $p \in \text{ap}(s)$ , with  $p \in \text{AP}$ .
- (2) For all formulas  $\varphi$ ,  $\varphi_1$ , and  $\varphi_2$ , it holds that:
  - (a)  $\mathcal{G}, \chi, s \models \neg\varphi$  if not  $\mathcal{G}, \chi, s \models \varphi$ , that is  $\mathcal{G}, \chi, s \not\models \varphi$ ;
  - (b)  $\mathcal{G}, \chi, s \models \varphi_1 \wedge \varphi_2$  if  $\mathcal{G}, \chi, s \models \varphi_1$  and  $\mathcal{G}, \chi, s \models \varphi_2$ ;
  - (c)  $\mathcal{G}, \chi, s \models \varphi_1 \vee \varphi_2$  if  $\mathcal{G}, \chi, s \models \varphi_1$  or  $\mathcal{G}, \chi, s \models \varphi_2$ .
- (3) For a variable  $x \in \text{Vr}$  and a formula  $\varphi$ , it holds that:
  - (a)  $\mathcal{G}, \chi, s \models \langle\langle x \rangle\rangle\varphi$  if there is a strategy  $f \in \text{Str}$  such that  $\mathcal{G}, \chi[x \mapsto f], s \models \varphi$ ;
  - (b)  $\mathcal{G}, \chi, s \models \llbracket x \rrbracket\varphi$  if, for all strategies  $f \in \text{Str}$ , it holds that  $\mathcal{G}, \chi[x \mapsto f], s \models \varphi$ .
- (4) For an agent  $a \in \text{Ag}$ , a variable  $x \in \text{Vr}$ , and a formula  $\varphi$ , it holds that  $\mathcal{G}, \chi, s \models (a, x)\varphi$  if  $\mathcal{G}, \chi[a \mapsto \chi(x)], s \models \varphi$ .
- (5) Finally, if the assignment  $\chi$  is complete, for all formulas  $\varphi$ ,  $\varphi_1$ , and  $\varphi_2$ , it holds that:
  - (a)  $\mathcal{G}, \chi, s \models \mathbf{X}\varphi$  if  $\mathcal{G}, (\chi, s)^1 \models \varphi$ ;
  - (b)  $\mathcal{G}, \chi, s \models \varphi_1 \mathbf{U} \varphi_2$  if there is an index  $i \in \mathbb{N}$  with  $k \leq i$  such that  $\mathcal{G}, (\chi, s)^i \models \varphi_2$  and, for all indexes  $j \in \mathbb{N}$  with  $k \leq j < i$ , it holds that  $\mathcal{G}, (\chi, s)^j \models \varphi_1$ ;
  - (c)  $\mathcal{G}, \chi, s \models \varphi_1 \mathbf{R} \varphi_2$  if, for all indexes  $i \in \mathbb{N}$  with  $k \leq i$ , it holds that  $\mathcal{G}, (\chi, s)^i \models \varphi_2$  or there is an index  $j \in \mathbb{N}$  with  $k \leq j < i$  such that  $\mathcal{G}, (\chi, s)^j \models \varphi_1$ .

Intuitively, at Items 3a and 3b, respectively, we evaluate the existential  $\langle\langle x \rangle\rangle$  and universal  $\llbracket x \rrbracket$  quantifiers over strategies, by associating them to the variable  $x$ . Moreover, at Item 4, by means of an agent binding  $(a, x)$ , we commit the agent  $a$  to a strategy associated with the variable  $x$ . It is evident that, due to Items 5a, 5b, and 5c, the LTL semantics is simply embedded into the SL one.

In order to complete the description of the semantics, we now give the classic notions of *model* and *satisfiability* of an SL sentence. We say that a CGS  $\mathcal{G}$  is a *model* of an SL sentence  $\varphi$ , in symbols  $\mathcal{G} \models \varphi$ , if  $\mathcal{G}, \emptyset, s_0 \models \varphi$ .<sup>8</sup> In general, we also say that  $\mathcal{G}$  is a *model* for  $\varphi$  on  $s \in \text{St}$ , in symbols  $\mathcal{G}, s \models \varphi$ , if  $\mathcal{G}, \emptyset, s \models \varphi$ . An SL sentence  $\varphi$  is *satisfiable* if there is a model for it.

It remains to formalize the concepts of *implication* and *equivalence* between SL formulas, which are useful to describe transformations preserving the meaning of a specification.

<sup>8</sup>The symbol  $\emptyset$  stands for the empty function.

Given two SL formulas  $\varphi_1$  and  $\varphi_2$ , with  $\text{free}(\varphi_1) = \text{free}(\varphi_2)$ , we say that  $\varphi_1$  *implies*  $\varphi_2$ , in symbols  $\varphi_1 \Rightarrow \varphi_2$ , if, for all CGSs  $\mathcal{G}$ , states  $s \in \text{St}$ , and  $\text{free}(\varphi_1)$ -defined assignments  $\chi \in \text{Asg}(\text{free}(\varphi_1), s)$ , it holds that if  $\mathcal{G}, \chi, s \models \varphi_1$  then  $\mathcal{G}, \chi, s \models \varphi_2$ . Accordingly, we say that  $\varphi_1$  is *equivalent* to  $\varphi_2$ , in symbols  $\varphi_1 \equiv \varphi_2$ , if both  $\varphi_1 \Rightarrow \varphi_2$  and  $\varphi_2 \Rightarrow \varphi_1$  hold.

In the rest of the paper, especially when we describe a decision procedure, we may consider formulas in *existential normal form* (*enf*, for short) and *positive normal form* (*pnf*, for short), *i.e.*, formulas in which only existential quantifiers appear or, respectively, the negation is applied solely to atomic propositions. In fact, it is to this aim that we have considered in the syntax of SL both the Boolean connectives  $\wedge$  and  $\vee$ , the temporal operators  $\mathbf{U}$ , and  $\mathbf{R}$ , and the strategy quantifiers  $\langle\langle \cdot \rangle\rangle$  and  $\llbracket \cdot \rrbracket$ . Indeed, all formulas can be linearly translated in *enf* and *pnf* by using De Morgan's laws together with the following equivalences, which directly follow from the semantics of the logic:  $\neg \mathbf{X}\varphi \equiv \mathbf{X}\neg\varphi$ ,  $\neg(\varphi_1 \mathbf{U} \varphi_2) \equiv (\neg\varphi_1) \mathbf{R} (\neg\varphi_2)$ ,  $\neg \langle\langle x \rangle\rangle \varphi \equiv \llbracket x \rrbracket \neg\varphi$ , and  $\neg(a, x)\varphi \equiv (a, x)\neg\varphi$ .

At this point, in order to better understand the meaning of the SL semantics, we discuss some examples of formulas interpreted over the CGSs previously described.

**Example 2.15** (Law and Order). Consider the CGS  $\mathcal{G}_{PPD}$  given in Example 2.3. It is easy to see that Police can ensure at least one prisoner to never be free. Indeed the formula  $\varphi_1 = \langle\langle y \rangle\rangle \llbracket x_1 \rrbracket \llbracket x_2 \rrbracket (\mathbf{P}, y)(\mathbf{A}_1, x_1)(\mathbf{A}_2, x_2)((\mathbf{FG}\neg\mathbf{f}_{\mathbf{A}_1}) \vee (\mathbf{FG}\neg\mathbf{f}_{\mathbf{A}_2}))$  is satisfied over  $\mathcal{G}_{PPD}$ . A way to see this is to consider the strategy  $\mathbf{f}$  for  $\mathbf{P}$  given by  $\mathbf{f}(\rho) = 0$ , for all  $\rho \in \text{Trk}$ , which allows agent  $\mathbf{P}$  to always avoid the state  $s_{\mathbf{A}_1, \mathbf{A}_2}$ . On the other hand, the formula  $\varphi_2 = \llbracket x_1 \rrbracket \llbracket x_2 \rrbracket \langle\langle y \rangle\rangle (\mathbf{P}, y)(\mathbf{A}_1, x_1)(\mathbf{A}_2, x_2)(\mathbf{FG}(\neg\mathbf{f}_{\mathbf{A}_1} \wedge \neg\mathbf{f}_{\mathbf{A}_2}))$  is not satisfied over  $\mathcal{G}_{PPD}$ . Indeed, if agents  $\mathbf{A}_1$  and  $\mathbf{A}_2$  use strategies  $\mathbf{f}_1$  and  $\mathbf{f}_2$ , respectively, such that  $\mathbf{f}_2(s_i) = 1 - \mathbf{f}_1(s_i)$ , we have that, whatever agent  $\mathbf{P}$  does, at least one of them gains freedom.

**Example 2.16** (Fair Scheduler). Consider the CGS  $\mathcal{G}_{PS}$  of Example 2.4 and suppose the Scheduler wants to ensure that, whatever process makes a request, the resource is eventually granted to it. We can represent this specification by means of the formula  $\varphi = \langle\langle y \rangle\rangle \llbracket x_1 \rrbracket \llbracket x_2 \rrbracket (\mathbf{S}, y)(\mathbf{P}_1, x_1)(\mathbf{P}_2, x_2)(\mathbf{G}((\mathbf{r}_1 \rightarrow \mathbf{Fg}_1) \wedge (\mathbf{r}_2 \rightarrow \mathbf{Fg}_2)))$ . It is easy to see that  $\mathcal{G}_{PS} \models \varphi$ . Indeed, consider the strategy  $\mathbf{f}$  for  $\mathbf{S}$  defined as follows. For all tracks of the form  $\rho \cdot s'_i$ , we set a possible preemption, *i.e.*,  $\mathbf{f}(\rho \cdot s'_i) = 1$ . For the tracks of the form  $\rho \cdot s_{1,2}$  we set the action as prescribed in the sequel: (i) if there is no occurrence of  $s'_1$  and  $s'_2$  or the last occurrence is  $s'_1$ , then we release the resource to  $\mathbf{P}_1$ , *i.e.*,  $\mathbf{f}(\rho \cdot s_{1,2}) = 0$ ; (ii) if the last occurrence in  $\rho$  between  $s'_1$  and  $s'_2$  is  $s'_2$ , then then we release the resource to  $\mathbf{P}_2$ , *i.e.*,  $\mathbf{f}(\rho \cdot s_{1,2}) = 1$ .

### 3. HARDNESS RESULTS

In [MMPV14] it has been shown that the model-checking for SL is NONELEMENTARY-HARD. Here, we prove that the satisfiability problem is even harder, *i.e.*, undecidable. To do this, we first introduce a sentence that is satisfiable only on unbounded models. Then, by using this result, we prove the undecidability result through a reduction of the classic domino problem [Wan61].

**3.1. Unbounded models.** We now show that SL does not enjoy the bounded-tree model property. In general, a classic modal logic satisfies this property if, whenever a formula is satisfiable, it is so on a model in which all states have a number of successors bounded by an a priori fixed constant. However, in the case of SL, this condition is not sufficient to characterize bounded models, since SL has the power of distinguishing among different ways to reach a given state from another one, and the number of ways might be infinite, as that of actions can be infinite itself. An example of an unbounded model with a finite number of states is given in Figure 3. For this reason, we say that a CGS is *bounded* if the set of actions  $\text{Ac}_{\mathcal{G}}$  is finite. Moreover, a model is *finite* if it is bounded and the number of states is also finite. Clearly, if a logic invariant under unwinding enjoys the finite model property, it enjoys the bounded-tree model property as well. The other direction may not hold, instead, as exemplified by the  $\mu\text{CALCULUS}$  with backward modalities [Var98, Boj03]. Unfortunately, SL does not enjoy either property.

In order to prove the existence of satisfiable SL formulas with unbounded models only, we introduce, in the following definition, the sentence  $\varphi^{\text{ord}}$  to be used as a counterexample for the bounded-tree model property.

**Definition 3.1** (Ordering Sentence). Let  $x_1 < x_2 \triangleq \langle\langle y \rangle\rangle \varphi(x_1, x_2, y)$  be an agent-closed formula, named *partial order*, on the sets  $\text{AP} = \{p\}$  and  $\text{Ag} = \{\alpha, \beta\}$ , where  $\varphi(x_1, x_2, y) \triangleq ((\alpha, x_1)(\beta, y)(Xp)) \wedge ((\alpha, x_2)(\beta, y)(X\neg p))$ . Then, the *ordering sentence*  $\varphi^{\text{ord}} \triangleq \varphi^{\text{unb}} \wedge \varphi^{\text{trn}}$  is the conjunction of the following two sentences, called *unboundedness* and *transitivity* strategy requirements:

- (1)  $\varphi^{\text{unb}} \triangleq \langle\langle x_1 \rangle\rangle \langle\langle x_2 \rangle\rangle x_1 < x_2$ ;
- (2)  $\varphi^{\text{trn}} \triangleq \langle\langle x_1 \rangle\rangle \langle\langle x_2 \rangle\rangle \langle\langle x_3 \rangle\rangle (x_1 < x_2 \wedge x_2 < x_3) \rightarrow x_1 < x_3$ .

Intuitively,  $\varphi^{\text{unb}}$  asserts that, for each strategy in  $x_1$ , there is a different strategy in  $x_2$  that is in relation of  $<$  w.r.t. the first one, i.e.,  $<$  has no upper bound, due to the fact that, by the definition of  $\varphi(x_1, x_2, y)$ , it is not reflexive. Moreover,  $\varphi^{\text{trn}}$  ensures that the relation  $<$  is transitive too. Consequently,  $\varphi^{\text{ord}}$  induces a strict partial pre-order on the strategies.

Obviously, in order to be useful, the sentence  $\varphi^{\text{ord}}$  needs to be satisfiable, as reported in the following lemma.

**Lemma 3.2** (Ordering Satisfiability). *The sentence  $\varphi^{\text{ord}}$  is satisfiable.*

*Proof.* To prove that  $\varphi^{\text{ord}}$  is satisfiable, consider the unbounded CGS  $\mathcal{G}^*$  in Figure 3, where (i)  $\text{AP} \triangleq \{p\}$ , (ii)  $\text{Ag} \triangleq \{\alpha, \beta\}$ , (iii)  $\text{Ac}_{\mathcal{G}^*} \triangleq \mathbb{N}$ , (iv)  $\text{St}_{\mathcal{G}^*} \triangleq \{s_0, s_1, s_2\}$ , (v)  $s_0_{\mathcal{G}^*} = s_0$ , (vi)  $\text{ap}_{\mathcal{G}^*}(s_0) = \text{ap}_{\mathcal{G}^*}(s_2) \triangleq \emptyset$  and  $\text{ap}_{\mathcal{G}^*}(s_1) \triangleq \{p\}$ , (vii)  $\text{P} \triangleq \{\delta \in \text{Dc}_{\mathcal{G}^*} : \delta(\alpha) \leq \delta(\beta)\}$ , and (viii)  $\text{tr}_{\mathcal{G}^*}$  is such that if  $\delta \in \text{P}$  then  $\text{tr}_{\mathcal{G}^*}(s_0, \delta) = s_1$  else  $\text{tr}_{\mathcal{G}^*}(s_0, \delta) = s_2$ , and  $\text{tr}_{\mathcal{G}^*}(s, \delta) = s$ , for all  $s \in \{s_1, s_2\}$  and  $\delta \in \text{Dc}_{\mathcal{G}^*}$ .

Now, it is easy to see that  $\mathcal{G}^* \models \varphi^{\text{unb}}$ , since for every strategy  $f_{x_1}$  for  $x_1$ , consisting of picking a natural number  $n = f_{x_1}(s_0)$  as an action at the initial state, we can reply with a strategy  $f_{x_2}$  for  $x_2$  having  $f_{x_2}(s_0) > n$  and a strategy  $f_y$  for  $y$  having  $f_y(s_0) = n$ . Formally, we have that  $\mathcal{G}^*, \chi, s_0 \models \varphi(x_1, x_2, y)$  iff  $\chi(x_1)(s_0) \leq \chi(y)(s_0) < \chi(x_2)(s_0)$ , for all assignments  $\chi \in \text{Asg}_{\mathcal{G}^*}(\{x_1, x_2, y\}, s_0)$ .

By a similar reasoning, we can see that  $\mathcal{G}^* \models \varphi^{\text{trn}}$ . Indeed, consider three strategies  $f_{x_1}$ ,  $f_{x_2}$ , and  $f_{x_3}$  for the variables  $x_1$ ,  $x_2$ , and  $x_3$ , respectively, which correspond to picking three

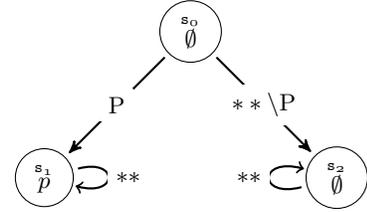


Figure 3: The CGS  $\mathcal{G}^*$  model of  $\varphi^{\text{ord}}$ .

natural numbers  $n_1 = f_{x_1}(s_0)$ ,  $n_2 = f_{x_2}(s_0)$ , and  $n_3 = f_{x_3}(s_0)$ . Now, if  $\mathcal{G}^*, \chi, s_0 \models x_1 < x_2$  and  $\mathcal{G}^*, \chi, s_0 \models x_2 < x_3$ , for an assignments  $\chi \in \text{Asg}_{\mathcal{G}^*}(\{x_1, x_2, x_3\}, s_0)$  where  $\chi(x_1) = f_{x_1}$ ,  $\chi(x_2) = f_{x_2}$ , and  $\chi(x_3) = f_{x_3}$ , we have that  $n_1 < n_2$  and  $n_2 < n_3$ . Consequently,  $n_1 < n_3$ . Hence, by using a strategy  $f_y$  for  $y$  with  $f_y(s_0) = f_{x_1}(s_0)$ , we have  $\mathcal{G}^*, \chi_{y \rightarrow f_y}, s_0 \models \varphi(x_1, x_3, y)$  and thus  $\mathcal{G}^*, \chi, s_0 \models x_1 < x_3$ .  $\square$

Next lemmas report two important properties of the sentence  $\varphi^{ord}$ , for the negative statements we want to show. Namely, they state that, in order to be satisfied,  $\varphi^{ord}$  must require the existence of strict partial order relations on strategies and actions that do not admit any maximal element. From this, as stated in Theorem 3.5, we directly derive that  $\varphi^{ord}$  needs an infinite chain of actions to be satisfied, *i.e.*, it cannot have a bounded model.

**Lemma 3.3** (Strategy Order). *Let  $\mathcal{G}$  be a model of  $\varphi^{ord}$ . Moreover, let  $r^< \subseteq \text{Str}_{\mathcal{G}} \times \text{Str}_{\mathcal{G}}$  be a relation between strategies of  $\mathcal{G}$  such that  $r^<(f_1, f_2)$  holds iff  $\mathcal{G}, \emptyset[x_1 \mapsto f_1][x_2 \mapsto f_2], s_{0\mathcal{G}} \models x_1 < x_2$ , for all strategies  $f_1, f_2 \in \text{Str}_{\mathcal{G}}$ . Then,  $r^<$  is a strict partial order without maximal element.*

*Proof.* The proof derives from the fact that  $r^<$  satisfies the following properties:

- (1) *Irreflexivity:*  $\forall f \in \text{Str}_{\mathcal{G}}. \neg r^<(f, f)$ ;
- (2) *Unboundedness:*  $\forall f_1 \in \text{Str}_{\mathcal{G}} \exists f_2 \in \text{Str}_{\mathcal{G}}. r^<(f_1, f_2)$ ;
- (3) *Transitivity:*  $\forall f_1, f_2, f_3 \in \text{Str}_{\mathcal{G}}. (r^<(f_1, f_2) \wedge r^<(f_2, f_3)) \rightarrow r^<(f_1, f_3)$ .

Indeed, Items (ii) and (iii) are directly derived from the strategy unboundedness and transitivity requirements. The proof of Item (i) derives, instead, from the following reasoning. By contradiction, suppose that  $r^<$  is not a strict order, *i.e.*, there is a strategy  $f \in \text{Str}_{\mathcal{G}}$  for which  $r^<(f, f)$  holds. This means that, at the initial state  $s_{0\mathcal{G}}$  of  $\mathcal{G}$ , there exists an assignment  $\chi \in \text{Asg}_{\mathcal{G}}(\{x_1, x_2, y\}, s_{0\mathcal{G}})$  for which  $\mathcal{G}, \chi, s_{0\mathcal{G}} \models \varphi(x_1, x_2, y)$ , where  $\chi(x_1) = \chi(x_2) = f$ . The last fact implies the existence of a successor of  $s_{0\mathcal{G}}$  in which both  $p$  and  $\neg p$  hold, which is clearly impossible.  $\square$

**Lemma 3.4** (Action Order). *Let  $\mathcal{G}$  be a model of  $\varphi^{ord}$ . Moreover, let  $s^< \subseteq \text{Ac}_{\mathcal{G}} \times \text{Ac}_{\mathcal{G}}$  be a relation between actions of  $\mathcal{G}$  such that  $s^<(c_1, c_2)$  holds iff, for all strategies  $f_1, f_2 \in \text{Str}_{\mathcal{G}}$  with  $c_1 = f_1(s_{0\mathcal{G}})$  and  $c_2 = f_2(s_{0\mathcal{G}})$ , it holds that  $r^<(f_1, f_2)$ , where  $c_1, c_2 \in \text{Ac}_{\mathcal{G}}$ . Then,  $s^<$  is a strict partial order without maximal element.*

*Proof.* The irreflexivity and transitivity of  $s^<$  are directly derived from the fact that, by Lemma 3.3,  $r^<$  is irreflexive and transitive too. The proof of the unboundedness property derives, instead, from the following reasoning. As first thing, observe that, since the formula  $x_1 < x_2$  relies on  $Xp$  and  $X\neg p$  as the only temporal operators, it holds that  $r^<(f_1, f_2)$  implies  $r^<(f'_1, f'_2)$ , for all strategies  $f_1, f_2, f'_1, f'_2 \in \text{Str}_{\mathcal{G}}$  such that  $f_1(s_{0\mathcal{G}}) = f'_1(s_{0\mathcal{G}})$  and  $f_2(s_{0\mathcal{G}}) = f'_2(s_{0\mathcal{G}})$ . Now, suppose by contradiction that  $s^<$  does not satisfy the unboundedness property, *i.e.*, there is an action  $c \in \text{Ac}_{\mathcal{G}}$  such that, for all actions  $c' \in \text{Ac}_{\mathcal{G}}$ , it does not hold that  $s^<(c, c')$ . Then, by the definition of  $s^<$  and the previous observation, we derive the existence of a strategy  $f \in \text{Str}_{\mathcal{G}}$  with  $f(s_{0\mathcal{G}}) = c$  such that  $r^<(f, f')$  does not hold, for any strategy  $f' \in \text{Str}_{\mathcal{G}}$ , which is clearly impossible.  $\square$

Now, we have all tools to prove that SL lacks of the bounded-tree model property, which hold, instead, for several commonly used multi-agent logics, such as ATL<sup>\*</sup>.

**Theorem 3.5** (SL Unbounded Model Property). *SL does not enjoy the bounded model property.*

*Proof.* To prove the statement, we show that the sentence  $\varphi^{ord}$  of Definition 3.1 cannot be satisfied on a bounded CGS. Consider a CGS  $\mathcal{G}$  such that  $\mathcal{G} \models \varphi^{ord}$ . The existence of such a model is ensured by Lemma 3.2. Now, consider the strict partial order without maximal element between actions  $s^<$  described in Lemma 3.4. By a classical result on first order logic model theory [EF95], the relation  $s^<$  cannot be defined on a finite set. Hence,  $|\text{Ac}| = \infty$ .  $\square$

**3.2. Undecidable satisfiability.** We finally show the undecidability of the satisfiability problem for SL through a reduction from the *recurrent domino problem*.

The *domino problem*, proposed for the first time by Wang [Wan61], consists of placing a given number of tile types on an infinite grid, satisfying a predetermined set of constraints on adjacent tiles. One of its standard versions asks for a compatible tiling of the whole plane  $\mathbb{N} \times \mathbb{N}$ . The *recurrent domino problem* further requires the existence of a distinguished tile type that occurs infinitely often in the first row of the grid. This problem was proved to be highly undecidable by Harel, and in particular,  $\Sigma_1^1$ -COMPLETE [Har84]. The formal definition follows.

**Definition 3.6** (Recurrent Domino System). An  $\mathbb{N} \times \mathbb{N}$  *recurrent domino system*  $\mathcal{D} = \langle D, H, V, t_0 \rangle$  consists of a finite non-empty set  $D$  of *domino types*, two *horizontal* and *vertical matching relations*  $H, V \subseteq D \times D$ , and a *distinguished tile type*  $t_0 \in D$ . The recurrent domino problem asks for an *admissible tiling* of  $\mathbb{N} \times \mathbb{N}$ , which is a *solution mapping*  $\partial : \mathbb{N} \times \mathbb{N} \rightarrow D$  such that, for all  $x, y \in \mathbb{N}$ , it holds that (i)  $(\partial(x, y), \partial(x + 1, y)) \in H$ , (ii)  $(\partial(x, y), \partial(x, y + 1)) \in V$ , and (iii)  $|\{x \in \mathbb{N} : \partial(x, 0) = t_0\}| = \omega$ .

**Grid specification.** Consider the sentence  $\varphi^{grd} \triangleq \bigwedge_{a \in \text{Ag}} \varphi_a^{ord}$ , where  $\varphi_a^{ord} = \varphi_a^{unb} \wedge \varphi_a^{trn}$  are the *order sentences* and  $\varphi_a^{unb}$  and  $\varphi_a^{trn}$  are the *unboundedness* and *transitivity* strategy requirements for agents  $\alpha$  and  $\beta$  defined, similarly to Definition 3.1, as follows:

- (1)  $\varphi_a^{unb} \triangleq \llbracket z_1 \rrbracket \langle\langle z_2 \rangle\rangle z_1 <_a z_2$ ;
- (2)  $\varphi_a^{trn} \triangleq \llbracket z_1 \rrbracket \llbracket z_2 \rrbracket \llbracket z_3 \rrbracket (z_1 <_a z_2 \wedge z_2 <_a z_3) \rightarrow z_1 <_a z_3$ ;

where  $x_1 <_\alpha x_2 \triangleq \langle\langle y \rangle\rangle \varphi_\alpha(x_1, x_2, y)$  and  $y_1 <_\beta y_2 \triangleq \langle\langle x \rangle\rangle \varphi_\beta(y_1, y_2, x)$  are the two *partial order* formulas on strategies of  $\alpha$  and  $\beta$ , respectively, with  $\varphi_\alpha(x_1, x_2, y) \triangleq (\beta, y)((\alpha, x_1)(\mathbf{X}p) \wedge (\alpha, x_2)(\mathbf{X}\neg p))$  and  $\varphi_\beta(y_1, y_2, x) \triangleq (\alpha, x)((\beta, y_1)(\mathbf{X}\neg p) \wedge (\beta, y_2)(\mathbf{X}p))$ . Intuitively,  $<_\alpha$  and  $<_\beta$  correspond to the horizontal and vertical ordering of the positions in the grid, respectively.

It is easy to show that  $\varphi^{grd}$  is satisfiable, by using the same candidate model  $\mathcal{G}^*$  (see Figure 3) and a proof argument similar to that proposed in Lemma 3.2 for the simpler order sentence.

**Lemma 3.7** (Grid Ordering Satisfiability). *The sentence  $\varphi^{grd}$  is satisfiable.*

*Proof.* Let  $\mathcal{G}^*$  be the model described in Lemma 3.2. On one hand, by using the same lemma, it is evident that  $\mathcal{G}^* \models \varphi_\alpha^{ord}$ . On the other hand, in order to prove that  $\mathcal{G}^* \models \varphi_\beta^{ord}$ , first observe that  $\mathcal{G}^*, \chi, s_{0\mathcal{G}^*} \models \varphi_\beta(y_1, y_2, x)$  iff  $\chi(y_1)(s_{0\mathcal{G}^*}) < \chi(x)(s_{0\mathcal{G}^*}) \leq \chi(y_2)(s_{0\mathcal{G}^*})$ , for all assignments  $\chi \in \text{Asg}_{\mathcal{G}^*}(\{y_1, y_2, x\}, s_{0\mathcal{G}^*})$ . At this point, the thesis follows by a reasoning similar to the one proposed in the lemma.  $\square$

Consider now a model  $\mathcal{G}$  of  $\varphi^{grd}$  and, for all agents  $a \in \text{Ag}$ , the relation  $r_a^< \subseteq \text{Str}_{\mathcal{G}} \times \text{Str}_{\mathcal{G}}$  between strategies defined as follows:  $r_a^<(\mathbf{f}_1, \mathbf{f}_2)$  holds iff  $\mathcal{G}, \emptyset[z_1 \mapsto \mathbf{f}_1][z_2 \mapsto \mathbf{f}_2], s_{0\mathcal{G}} \models z_1 <_a z_2$ , for all strategies  $\mathbf{f}_1, \mathbf{f}_2 \in \text{Str}_{\mathcal{G}}$ . By using a proof similar to that of Lemma 3.3, it is possible to see that  $r_a^<$  is a *strict partial order without maximal element* on  $\text{Str}_{\mathcal{G}}$ .

Now, to apply the desired reduction, we need to transform  $r_a^<$  into a total order over strategies, by using the following two lemmas.

**Lemma 3.8** (Strategy Equivalence). *Let  $r_a^{\equiv} \subseteq \text{Str}_{\mathcal{G}} \times \text{Str}_{\mathcal{G}}$ , with  $a \in \text{Ag}$ , be the relation between strategies such that  $r_a^{\equiv}(\mathbf{f}_1, \mathbf{f}_2)$  holds iff neither  $r_a^<(\mathbf{f}_1, \mathbf{f}_2)$  nor  $r_a^<(\mathbf{f}_2, \mathbf{f}_1)$  holds, for all  $\mathbf{f}_1, \mathbf{f}_2 \in \text{Str}_{\mathcal{G}}$ . Then  $r_a^{\equiv}$  is an equivalence relation.*

*Proof.* It is immediate to see that the relation  $r_a^{\equiv}$  is reflexive, since  $r_a^<$  is not reflexive. Moreover, it is symmetric by definition. Finally, due to the definition of the partial order formula  $<_a$ , it is also transitive and, thus,  $r_a^{\equiv}$  is an *equivalence relation*. Indeed, if  $r_a^{\equiv}(\mathbf{f}_1, \mathbf{f}_2)$  holds, we have that  $\mathcal{G}, \chi_{12}, s_{0\mathcal{G}} \models \llbracket y \rrbracket(\beta, y)((\alpha, x_1)(\mathbf{X}\neg p) \vee (\alpha, x_2)(\mathbf{X}p))$  and  $\mathcal{G}, \chi_{12}, s_{0\mathcal{G}} \models \llbracket y \rrbracket(\beta, y)((\alpha, x_2)(\mathbf{X}\neg p) \vee (\alpha, x_1)(\mathbf{X}p))$ , for all assignments  $\chi_{12} \in \text{Asg}(\mathcal{G}, \{x_1, x_2\}, s_{0\mathcal{G}})$  such that  $\chi_{12}(x_1) = \mathbf{f}_1$  and  $\chi_{12}(x_2) = \mathbf{f}_2$ . Consequently,  $\mathcal{G}, \chi_{12}, s_{0\mathcal{G}} \models \llbracket y \rrbracket(\beta, y)((\alpha, x_1)(\mathbf{X}\neg p) \vee (\alpha, x_2)(\mathbf{X}p)) \wedge ((\alpha, x_2)(\mathbf{X}\neg p) \vee (\alpha, x_1)(\mathbf{X}p))$ , which is equivalent to  $\mathcal{G}, \chi_{12}, s_{0\mathcal{G}} \models \llbracket y \rrbracket(\beta, y)((\alpha, x_1)(\mathbf{X}p) \wedge (\alpha, x_2)(\mathbf{X}p)) \vee ((\alpha, x_1)(\mathbf{X}\neg p) \vee (\alpha, x_2)(\mathbf{X}\neg p))$ . In other words, for all strategies  $\mathbf{f}$ , either  $\mathcal{G}, \chi_{12}[y \mapsto \mathbf{f}], s_{0\mathcal{G}} \models (\beta, y)((\alpha, x_1)(\mathbf{X}p) \wedge (\alpha, x_2)(\mathbf{X}p))$  or  $\mathcal{G}, \chi_{12}[y \mapsto \mathbf{f}], s_{0\mathcal{G}} \models (\beta, y)((\alpha, x_1)(\mathbf{X}\neg p) \wedge (\alpha, x_2)(\mathbf{X}\neg p))$  holds. Similarly, from  $r_a^{\equiv}(\mathbf{f}_2, \mathbf{f}_3)$ , we can derive that, for all strategies  $\mathbf{f}$  and assignments  $\chi_{23} \in \text{Asg}(\mathcal{G}, \{x_2, x_3\}, s_{0\mathcal{G}})$  with  $\chi_{23}(x_2) = \mathbf{f}_2$  and  $\chi_{23}(x_3) = \mathbf{f}_3$ , either  $\mathcal{G}, \chi_{23}[y \mapsto \mathbf{f}], s_{0\mathcal{G}} \models (\beta, y)((\alpha, x_2)(\mathbf{X}p) \wedge (\alpha, x_3)(\mathbf{X}p))$  or  $\mathcal{G}, \chi_{23}[y \mapsto \mathbf{f}], s_{0\mathcal{G}} \models (\beta, y)((\alpha, x_2)(\mathbf{X}\neg p) \wedge (\alpha, x_3)(\mathbf{X}\neg p))$  holds. Therefore, by putting the two deductions together, we have that either  $\mathcal{G}, \chi_{13}, s_{0\mathcal{G}} \models (\beta, y)((\alpha, x_1)(\mathbf{X}p) \wedge (\alpha, x_3)(\mathbf{X}p))$  or  $\mathcal{G}, \chi_{13}, s_{0\mathcal{G}} \models (\beta, y)((\alpha, x_1)(\mathbf{X}\neg p) \wedge (\alpha, x_3)(\mathbf{X}\neg p))$  holds, for all assignments  $\chi_{13} \in \text{Asg}(\mathcal{G}, \{x_1, x_3, y\}, s_{0\mathcal{G}})$  such that  $\chi_{13}(x_1) = \mathbf{f}_1$  and  $\chi_{13}(x_3) = \mathbf{f}_3$ . Thus, by following the above reasoning at the reverse, we immediately derive that  $r_a^{\equiv}(\mathbf{f}_1, \mathbf{f}_3)$  holds, as well. Obviously, the same reasoning applies to  $r_b^{\equiv}$ .  $\square$

Let  $\text{Str}_a^{\equiv} \triangleq (\text{Str}_{\mathcal{G}}/r_a^{\equiv})$  be the quotient set of  $\text{Str}_{\mathcal{G}}$  w.r.t.  $r_a^{\equiv}$ , for  $a \in \text{Ag}$ , i.e., the set of the related equivalence classes over strategies. Then, the following holds.

**Lemma 3.9** (Strategy Total Order). *Let  $s_a^< \subseteq \text{Str}_a^{\equiv} \times \text{Str}_a^{\equiv}$ , with  $a \in \text{Ag}$ , be the relation between classes of strategies such that  $s_a^<(\mathbf{F}_1, \mathbf{F}_2)$  holds iff  $r_a^<(\mathbf{f}_1, \mathbf{f}_2)$  holds, for all  $\mathbf{f}_1 \in \mathbf{F}_1$ ,  $\mathbf{f}_2 \in \mathbf{F}_2$ , and  $\mathbf{F}_1, \mathbf{F}_2 \in \text{Str}_a^{\equiv}$ . Then  $s_a^<$  is a strict total order with minimal element but no maximal element.*

*Proof.* The fact that  $s_a^<$  is a *strict partial order without maximal element* derives directly from the same property of  $r_a^<$ . In fact, due to the specific definition of the partial order formula  $<_a$ , if  $r_a^{\equiv}(\mathbf{f}', \mathbf{f}'')$  and  $r_a^<(\mathbf{f}', \mathbf{f})$  (resp.,  $r_a^<(\mathbf{f}, \mathbf{f}')$ ) hold, we obtain that  $r_a^<(\mathbf{f}'', \mathbf{f})$  (resp.,  $r_a^<(\mathbf{f}, \mathbf{f}'')$ ) holds as well. Indeed, as shown in the proof of Lemma 3.8,  $r_a^{\equiv}(\mathbf{f}', \mathbf{f}'')$  implies that, for all assignments  $\chi_{\equiv} \in \text{Asg}(\mathcal{G}, \{x', x'', y\}, s_{0\mathcal{G}})$  with  $\chi_{\equiv}(x') = \mathbf{f}'$  and  $\chi_{\equiv}(x'') = \mathbf{f}''$ , either  $\mathcal{G}, \chi_{\equiv}, s_{0\mathcal{G}} \models (\beta, y)((\alpha, x')(\mathbf{X}p) \wedge (\alpha, x'')(\mathbf{X}p))$  or  $\mathcal{G}, \chi_{\equiv}, s_{0\mathcal{G}} \models (\beta, y)((\alpha, x')(\mathbf{X}\neg p) \wedge (\alpha, x'')(\mathbf{X}\neg p))$  holds. Moreover,  $r_a^<(\mathbf{f}', \mathbf{f})$  (resp.,  $r_a^<(\mathbf{f}, \mathbf{f}')$ ) implies that, for all assignments  $\chi_{<} \in \text{Asg}(\mathcal{G}, \{x, x'\}, s_{0\mathcal{G}})$  with  $\chi_{<}(x) = \mathbf{f}$  and  $\chi_{<}(x') = \mathbf{f}'$ , there exists a strategy  $\mathbf{g}$  such that  $\mathcal{G}, \chi_{<}[y \mapsto \mathbf{g}], s_{0\mathcal{G}} \models (\beta, y)((\alpha, x')(\mathbf{X}\neg p) \wedge (\alpha, x)(\mathbf{X}p))$  (resp.,  $\mathcal{G}, \chi_{<}[y \mapsto \mathbf{g}], s_{0\mathcal{G}} \models (\beta, y)((\alpha, x)(\mathbf{X}\neg p) \wedge (\alpha, x')(\mathbf{X}p))$ ). Now, by combining both deductions w.r.t. the same strategy  $\mathbf{g}$  assigned to the variable  $y$ , we obtain that  $\mathcal{G}, \chi[y \mapsto \mathbf{g}], s_{0\mathcal{G}} \models (\beta, y)((\alpha, x'')(\mathbf{X}\neg p) \wedge (\alpha, x)(\mathbf{X}p))$  (resp.,  $\mathcal{G}, \chi[y \mapsto \mathbf{g}], s_{0\mathcal{G}} \models (\beta, y)((\alpha, x)(\mathbf{X}\neg p) \wedge (\alpha, x'')(\mathbf{X}p))$ ), for all assignments  $\chi \in \text{Asg}(\mathcal{G}, \{x, x''\}, s_{0\mathcal{G}})$  with  $\chi(x) = \mathbf{f}$  and  $\chi(x'') = \mathbf{f}''$ . Hence,  $r_a^<(\mathbf{f}'', \mathbf{f})$  (resp.,  $r_a^<(\mathbf{f}, \mathbf{f}'')$ ).

Obviously, the same reasoning applies if we assume  $a = \beta$  instead of  $a = \alpha$ . At this point, if there are  $f_1 \in F_1$  and  $f_2 \in F_2$  such that  $r_a^<(f_1, f_2)$  holds, we directly obtain that  $s_a^<(F_1, F_2)$  holds as well, for all  $F_1, F_2 \in \text{Str}_a^{\equiv}$  and  $a \in \text{Ag}$ .

Moreover,  $s_a^<$  is total, since  $r_a^{\equiv}$  is an equivalence relation that cluster together all strategies of the agent  $a$  that are not in relation *w.r.t.* either  $r_a^<$  or its inverse  $(r_a^<)^{-1}$ . Indeed, suppose by contradiction that there are two different classes  $F_1, F_2 \in \text{Str}_a^{\equiv}$  such that neither  $s_a^<(F_1, F_2)$  nor  $s_a^<(F_2, F_1)$  holds. This means that, for all  $f_1 \in F_1$  and  $f_2 \in F_2$ , neither  $r_a^<(f_1, f_2)$  nor  $r_a^<(f_2, f_1)$  holds and, so,  $r_a^{\equiv}(f_1, f_2)$ . But, this contradicts the fact that  $F_1$  and  $F_2$  are different equivalence classes.

Finally, it is important to note that in  $\text{Str}_a^{\equiv}$  there is also a minimal element *w.r.t.*  $s_a^<$ . Indeed, for a strategy  $f \in \text{Str}_{\mathcal{G}}$  for  $\alpha$  (resp., for  $\beta$ ) that forces the play to reach only nodes labeled with  $p$  (resp.,  $\neg p$ ) as successor of  $s_{0\mathcal{G}}$ , independently from the strategy of  $\beta$  (resp.,  $\alpha$ ), the relation  $r_a^<(f', f)$  (resp.,  $r_\beta^<(f', f)$ ) cannot hold, for any  $f' \in \text{Str}_{\mathcal{G}}$ .  $\square$

By a classical result on first-order model theory [EF95], the relation  $s_a^<$  cannot be defined on a finite set. Hence,  $|\text{Str}_a^{\equiv}| = \omega$ , for all  $a \in \text{Ag}$ . Now, let  $s_a^< \subseteq \text{Str}_a^{\equiv} \times \text{Str}_a^{\equiv}$  be the *successor* relation on  $\text{Str}_a^{\equiv}$  compatible with the strict total order  $s_a^<$ , *i.e.*, such that  $s_a^<(F_1, F_2)$  holds iff (i)  $s_a^<(F_1, F_2)$  holds and (ii) there is no  $F_3 \in \text{Str}_a^{\equiv}$  for which both  $s_a^<(F_1, F_3)$  and  $s_a^<(F_3, F_2)$  hold, for all  $F_1, F_2 \in \text{Str}_a^{\equiv}$ . Then, we can represent the two sets of classes  $\text{Str}_\alpha^{\equiv}$  and  $\text{Str}_\beta^{\equiv}$ , respectively, as the infinite ordered lists  $\{F_0^\alpha, F_1^\alpha, \dots\}$  and  $\{F_0^\beta, F_1^\beta, \dots\}$  such that  $s_a^<(F_i^a, F_{i+1}^a)$  holds, for all indexes  $i \in \mathbb{N}$ . Note that  $F_0^a$  is the class of minimal strategies w.r.t the relation  $s_a^<$ .

At this point, we have all the machinery to build an embedding of the plane  $\mathbb{N} \times \mathbb{N}$  into a model  $\mathcal{G}$  of  $\varphi^{grd}$ . Formally, we consider the *bijective map*  $\aleph : \mathbb{N} \times \mathbb{N} \rightarrow \text{Str}_\alpha^{\equiv} \times \text{Str}_\beta^{\equiv}$  such that  $\aleph(i, j) = (F_i^\alpha, F_j^\beta)$ , for all  $i, j \in \mathbb{N}$ .

**Compatible tiling.** Given the grid structure built on the model  $\mathcal{G}$  of  $\varphi^{grd}$  through the bijective map  $\aleph$ , we can express that a tiling of the grid is admissible by making use of the formula  $z_1 \prec_a z_2 \triangleq (z_1 <_a z_2) \wedge (\neg \langle z_3 \rangle z_1 <_a z_3) \wedge (z_3 <_a z_2)$  corresponding to the successor relation  $s_a^<$ , for all  $a \in \text{Ag}$ . Indeed, it is not hard to see that  $\mathcal{G}, \chi, s_{0\mathcal{G}} \models z_1 \prec_a z_2$  iff  $\chi(z_1) \in F_i^a$  and  $\chi(z_2) \in F_{i+1}^a$ , for all indexes  $i \in \mathbb{N}$  and assignments  $\chi \in \text{Asg}_{\mathcal{G}}(\{z_1, z_2\}, s_{0\mathcal{G}})$ . The idea here is to associate with each domino type  $t \in \text{D}$  a corresponding atomic proposition  $t \in \text{AP}$  and to express the horizontal and vertical matching conditions via suitable object labeling. In particular, we can express that the tiling is locally compatible, the horizontal neighborhoods of a tile satisfy the  $H$  or  $V$  requirements, respectively. All these constraints can be formulated through the following three agent-closed formulas:

- (1)  $\varphi^{t,loc}(x, y) \triangleq (\alpha, x)(\beta, y)(\mathbf{X}(t \wedge \bigwedge_{t' \in \text{D}}^{t' \neq t} \neg t'))$ ;
- (2)  $\varphi^{t,hor}(x, y) \triangleq \bigvee_{(t, t') \in H} \llbracket x' \rrbracket (x \prec_\alpha x' \rightarrow (\alpha, x')(\beta, y)(\mathbf{X}t'))$ ;
- (3)  $\varphi^{t,ver}(x, y) \triangleq \bigvee_{(t, t') \in V} \llbracket y' \rrbracket (y \prec_\beta y' \rightarrow (\alpha, x)(\beta, y')(\mathbf{X}t'))$ .

Informally, we have the following:  $\varphi^{t,loc}(x, y)$  asserts that  $t$  is the only domino type labeling the successors of the root of the model  $\mathcal{G}$  that can be reached using the strategies related to the variables  $x$  and  $y$ ;  $\varphi^{t,hor}(x, y)$  asserts that the tile  $t'$  labeling the successors of the root reachable through the strategies  $x'$  and  $y$  is compatible with  $t$  *w.r.t.* the horizontal requirement  $H$ , for all strategies  $x'$  that immediately follow that related to  $x$  *w.r.t.* the order  $r_\alpha^<$ ;  $\varphi^{t,ver}(x, y)$  asserts that the tile  $t'$  labeling the successors of the root reachable

through the strategies  $x$  and  $y'$  is compatible with  $t$  *w.r.t.* the vertical requirement  $V$ , for all strategies  $y'$  that immediately follow that related to  $y$  *w.r.t.* the order  $r_\beta^<$ .

Finally, to express that the whole grid has an admissible tiling, we use the sentence  $\varphi^{til} \triangleq \llbracket x \rrbracket \llbracket y \rrbracket \bigvee_{t \in \mathcal{D}} \varphi^{t,loc}(x,y) \wedge \varphi^{t,hor}(x,y) \wedge \varphi^{t,ver}(x,y)$  that asserts the existence of a domino type  $t$  satisfying the three conditions mentioned above, for every point identified by the strategies  $x$  and  $y$ .

**Recurrent tile.** As last task, we impose that the grid embedded into  $\mathcal{G}$  has the distinguished domino type  $t_0$  occurring infinitely often in its first row. To do this, we describe two formulas that determine if a row or a column is the first one *w.r.t.* the orders  $s_\alpha^<$  and  $s_\beta^<$ , respectively. Formally, we use  $0_a(z) \triangleq \neg \langle \langle z' \rangle \rangle z' <_a z$ , for  $a \in \text{Ag}$ . One can easily prove that  $\mathcal{G}, \chi, s_{0\mathcal{G}} \models 0_a(z)$  iff  $\chi(z) \in F_0^a$ , for all assignments  $\chi \in \text{Asg}_{\mathcal{G}}(\{z\}, s_{0\mathcal{G}})$ . Now, the infinite occurrence requirement on  $t_0$  can be expressed with the following sentence:  $\varphi^{rec} \triangleq \llbracket x \rrbracket \llbracket y \rrbracket (0_\beta(y) \wedge (0_\alpha(x) \vee (\alpha, x)(\beta, y)(\mathbf{X}t_0))) \rightarrow \langle \langle x' \rangle \rangle x <_\alpha x' \wedge (\alpha, x')(\beta, y)(\mathbf{X}t_0)$ . Informally,  $\varphi^{rec}$  asserts that, when we are on the first row identified by the variable  $y$  and at a column pointed by  $x$  such that it is the first column or the node of the “*intersection*” between  $x$  and  $y$  is labeled by  $t_0$ , we have that there exists a greater column identified by  $x'$  such that its “*intersection*” with  $y$  is labeled by  $t_0$  as well.

**Construction correctness.** At this point, we have all tools to formally prove the correctness of the undecidability reduction, by showing the equivalence between the satisfiability of the sentence  $\varphi^{dom} \triangleq \varphi^{grd} \wedge \varphi^{til} \wedge \varphi^{rec}$  and finding a solution of the recurrent tiling problem.

**Theorem 3.10** (Satisfiability). *The satisfiability problem for SL is highly undecidable. In particular, it is  $\Sigma_1^1$ -HARD.*

*Proof.* For the direct reduction, assume that there exists a solution mapping  $\partial : \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{D}$  for the given recurrent domino system  $\mathcal{D}$ . Then, we can build a finite CGS  $\mathcal{G}_\partial^*$  similar to the one used in Lemma 3.2, which satisfies the sentence  $\varphi^{dom}$ :

- (i)  $\text{Ac}_{\mathcal{G}_\partial^*} \triangleq \mathbb{N}$ ;
- (ii) there are  $2 \cdot |\mathcal{D}| + 1$  different states  $\text{St}_{\mathcal{G}_\partial^*} \triangleq \{s_0\} \cup (\{p, \neg p\} \times \mathcal{D})$  such that  $\text{ap}_{\mathcal{G}_\partial^*}(s_0) \triangleq \emptyset$ ,  $\text{ap}_{\mathcal{G}_\partial^*}((p, t)) \triangleq \{p, t\}$ , and  $\text{ap}_{\mathcal{G}_\partial^*}((\neg p, t)) \triangleq \{t\}$ , for all  $t \in \mathcal{D}$ ;
- (iii) each state  $(z, t) \in \{p, \neg p\} \times \mathcal{D}$  has only self loops  $\text{tr}_{\mathcal{G}_\partial^*}((z, t), \delta) \triangleq (z, t)$  and the initial state  $s_{0\mathcal{G}_\partial^*} \triangleq s_0$  is connected to  $(z, t)$  through the decision  $\delta$ , *i.e.*,  $\text{tr}_{\mathcal{G}_\partial^*}(s_0, \delta) \triangleq (z, t)$ , iff
  - (a)  $t = \partial(\delta(\alpha), \delta(\beta))$  and
  - (b)  $z = p$  iff  $\delta(\alpha) \leq \delta(\beta)$ , for all  $\delta \in \text{Dc}_{\mathcal{G}_\partial^*}$ .

By a simple case analysis on the subformulas of  $\varphi^{dom}$ , it is possible to see that  $\mathcal{G}_\partial^* \models \varphi^{dom}$ .

Conversely, let  $\mathcal{G}$  be a model of the sentence  $\varphi^{dom}$  and  $\mathfrak{N} : \mathbb{N} \times \mathbb{N} \rightarrow \text{Str}_\alpha^= \times \text{Str}_\beta^=$  the related bijective map built for the grid specification task. As first thing, we have to prove

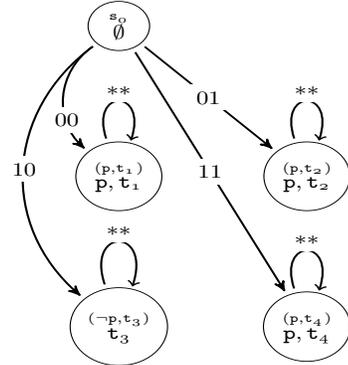


Figure 4: Part of the CGS  $\mathcal{G}_\partial^*$  model of  $\varphi^{dom}$ , where  $\partial(0, 0) = t_1$ ,  $\partial(0, 1) = t_2$ ,  $\partial(1, 0) = t_3$ , and  $\partial(1, 1) = t_4$ .

the existence of a coloring function  $\tilde{\partial} : \text{Str}_{\alpha}^{\equiv} \times \text{Str}_{\beta}^{\equiv} \rightarrow D$  such that, for all pairs of classes of strategies  $(F^{\alpha}, F^{\beta}) \in \text{Str}_{\alpha}^{\equiv} \times \text{Str}_{\beta}^{\equiv}$  and assignments  $\chi \in \text{Asg}_{\mathcal{G}}(\{\alpha, \beta\}, s_{0\mathcal{G}})$  with  $\chi(\alpha) \in F^{\alpha}$  and  $\chi(\beta) \in F^{\beta}$ , it holds that  $\mathcal{G}, \chi, s_{0\mathcal{G}} \models \mathbf{X}\tilde{\partial}(F^{\alpha}, F^{\beta})$ . Then, it remains to note that the solution mapping  $\partial = \tilde{\partial} \circ \aleph$  built as a composition of the bijective map  $\aleph$  and the coloring function  $\tilde{\partial}$  is an admissible tiling of the plane  $\mathbb{N} \times \mathbb{N}$ .

Due to the  $\varphi^{t,loc}$  formula in the sentence  $\varphi^{til}$ , we have that, for all assignments  $\chi \in \text{Asg}_{\mathcal{G}}(\{\alpha, \beta\}, s_{0\mathcal{G}})$ , there exists just one domino type  $t \in D$  satisfying the property  $\mathcal{G}, \chi, s_{0\mathcal{G}} \models \mathbf{X}t$ . Let  $\widehat{\partial} : \text{Str}_{\mathcal{G}} \times \text{Str}_{\mathcal{G}} \rightarrow D$  be the function that returns such a type, for all pairs of strategies of  $\alpha$  and  $\beta$ , *i.e.*, such that  $\mathcal{G}, \chi, s_{0\mathcal{G}} \models \mathbf{X}\widehat{\partial}(\chi(\alpha), \chi(\beta))$ , for all assignments  $\chi \in \text{Asg}_{\mathcal{G}}(\{\alpha, \beta\}, s_{0\mathcal{G}})$ . Now, it is not hard to see that, due to the formulas  $\varphi^{t,hor}$  and  $\varphi^{t,ver}$  in the sentence  $\varphi^{til}$ , it holds (i)  $(\widehat{\partial}(f_{\alpha}, f_{\beta}), \widehat{\partial}(f'_{\alpha}, f'_{\beta})) \in H$  and (ii)  $(\widehat{\partial}(f_{\alpha}, f_{\beta}), \widehat{\partial}(f_{\alpha}, f'_{\beta})) \in V$ , for all  $f_{\alpha} \in F_i^{\alpha}$ ,  $f'_{\alpha} \in F_{i+1}^{\alpha}$ ,  $f_{\beta} \in F_j^{\beta}$ ,  $f'_{\beta} \in F_{j+1}^{\beta}$ , and  $i, j \in \mathbb{N}$ . Moreover, the guess of the tile type  $t'$  adjacent to  $t$  is uniform *w.r.t.* the choice of the successor strategy. Indeed, the disjunctions  $\bigvee_{(t,t') \in H}$  and  $\bigvee_{(t,t') \in V}$  precede the universal quantifications  $\llbracket x' \rrbracket$  and  $\llbracket y' \rrbracket$  in the formulas  $\varphi^{t,hor}$  and  $\varphi^{t,ver}$ , respectively. Thus, we have that, for all  $f'_{\alpha}, f''_{\alpha} \in F_i^{\alpha}$  and  $f'_{\beta}, f''_{\beta} \in F_j^{\beta}$  with  $i, j \in \mathbb{N}$  and  $i + j > 0$ , it holds that  $\widehat{\partial}(f'_{\alpha}, f'_{\beta}) = \widehat{\partial}(f''_{\alpha}, f''_{\beta})$ . Note that this fact is not necessarily true for strategies belonging to the minimal classes  $F_0^{\alpha}$  and  $F_0^{\beta}$ , since the sentence  $\varphi^{dom}$  does not contain a relative requirement. However, every domino type  $\widehat{\partial}(f_{\alpha}, f_{\beta})$ , with  $f_{\alpha} \in F_0^{\alpha}$  and  $f_{\beta} \in F_0^{\beta}$ , can be used to label the origin of the plane  $\mathbb{N} \times \mathbb{N}$  in order to obtain an admissible tiling. So, we can consider a function  $\partial$ , defined as follows: (i)  $\partial(F_0^{\alpha}, F_0^{\beta}) \in \{\widehat{\partial}(f_{\alpha}, f_{\beta}) : f_{\alpha} \in F_0^{\alpha} \wedge f_{\beta} \in F_0^{\beta}\}$ ; (ii)  $\partial(F_i^{\alpha}, F_j^{\beta}) = \widehat{\partial}(f_{\alpha}, f_{\beta})$ , for all  $f_{\alpha} \in F_i^{\alpha}$ ,  $f_{\beta} \in F_j^{\beta}$ , and  $i, j \in \mathbb{N}$  with  $i + j > 0$ .

Clearly, (i)  $(\partial(F_i^{\alpha}, F_j^{\beta}), \partial(F_{i+1}^{\alpha}, F_j^{\beta})) \in H$ , (ii)  $(\partial(F_i^{\alpha}, F_j^{\beta}), \partial(F_i^{\alpha}, F_{j+1}^{\beta})) \in V$ , and (iii)  $|\{i : \partial(F_i^{\alpha}, F_0^{\beta}) = t_0\}| = \omega$ , for all  $i, j \in \mathbb{N}$ . So,  $\partial = \tilde{\partial} \circ \aleph$  is an admissible tiling.  $\square$

#### 4. WHAT MAKES ATL\* DECIDABLE?

As just shown, SL does not have the bounded model property and its satisfiability problem is undecidable. On the contrary, it is well-known that the satisfiability problem for ATL\* is 2EXPTIME-COMplete [Sch08]. This gap in complexity between SL and ATL\* gives naturally rise to the question of which are the inherent properties of ATL\* that make the problem decidable. In this section, we answer such question by analyzing two syntactic fragments of SL. The first one, called *Boolean-Goal Strategy Logic* (SL[BG], for short), still has an undecidable satisfiability problem. The second one, called *One-Goal Strategy Logic* (SL[1G], for short), retains, instead, all positive properties of ATL\*, such as the decision-tree model property (see Definition 4.12 and Theorem 4.13) and the bounded model property (see Theorem 5.4), which allows us to show that its satisfiability problem is 2EXPTIME-COMplete. A fundamental feature used as a tool to prove the announced properties is the *behavioral satisfiability*, described for the first time in [MMPV14].

The section is organized as follows. In Subsection 4.1, we introduce the syntactic fragments of SL mentioned above. Then, in Subsection 4.2, we define the concept of behavioral satisfiability and recall the corresponding theorem for SL[1G]. Finally, in Subsection 4.3, we introduce the concept of decision-tree model property and prove that it is enjoyed by the latter fragment.

**4.1. Syntactic fragments.** In order to formalize the two syntactic fragments of SL we want to investigate, we first need to define the concepts of *quantification* and *binding prefixes*. A *quantification prefix* over a set  $V \subseteq \text{Vr}$  of variables is a finite word  $\wp \in \{\langle\langle x \rangle\rangle, \llbracket x \rrbracket : x \in V\}^{|V|}$  of length  $|V|$  such that each variable  $x \in V$  occurs just once in  $\wp$ , *i.e.*, there is exactly one index  $i \in [0, |V|[$  such that  $(\wp)_i \in \{\langle\langle x \rangle\rangle, \llbracket x \rrbracket\}$ . A *binding prefix* over a set of variables  $V \subseteq \text{Vr}$  is a finite word  $\flat \in \{(a, x) : a \in \text{Ag} \wedge x \in V\}^{|\text{Ag}|}$  of length  $|\text{Ag}|$  such that each agent  $a \in \text{Ag}$  occurs just once in  $\flat$ , *i.e.*, there is exactly one index  $i \in [0, |\text{Ag}][$  for which  $(\flat)_i \in \{(a, x) : x \in V\}$ . By  $\text{Vr}(\wp)$  and  $\text{Vr}(\flat)$  we denote, respectively, the set of variables on which the quantification and binding prefixes  $\wp$  and  $\flat$  range. Finally,  $\text{Qnt}(V) \subseteq \{\langle\langle x \rangle\rangle, \llbracket x \rrbracket : x \in V\}^{|V|}$  and  $\text{Bnd}(V) \subseteq \{(a, x) : a \in \text{Ag} \wedge x \in V\}^{|\text{Ag}|}$  denote the sets of all quantification and binding prefixes over the variables in  $V$ .

We now have all tools to define the syntactic fragments named *Boolean-Goal* and *One-Goal Strategy Logic* ( $\text{SL}[\text{BG}]$  and  $\text{SL}[\text{1G}]$ , for short). For a *goal* we mean an SL agent-closed formula of the form  $\flat\varphi$ , with  $\text{Ag} \subseteq \text{free}(\varphi)$ , being  $\flat \in \text{Bnd}(\text{Vr})$  a binding prefix. The idea behind  $\text{SL}[\text{BG}]$  is to build sentences having only a Boolean combination of goals in the scope of a quantification prefix. Moreover,  $\text{SL}[\text{1G}]$  forces the use of a different quantification prefix for each goal in the formula. The formal syntax of  $\text{SL}[\text{BG}]$  and  $\text{SL}[\text{1G}]$  follows.

**Definition 4.1** ( $\text{SL}[\text{BG}]$  and  $\text{SL}[\text{1G}]$  Syntax).  $\text{SL}[\text{BG}]$  formulas are built inductively from the sets of atomic propositions AP, quantification prefixes  $\text{Qnt}(V)$  for any  $V \subseteq \text{Vr}$ , and binding prefixes  $\text{Bnd}(\text{Vr})$ , by using the following grammar, with  $p \in \text{AP}$ ,  $\wp \in \cup_{V \subseteq \text{Vr}} \text{Qnt}(V)$ , and  $\flat \in \text{Bnd}(\text{Vr})$ :

$$\begin{aligned} \varphi &::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{X}\varphi \mid \varphi \mathbf{U}\varphi \mid \varphi \mathbf{R}\varphi \mid \wp\psi, \\ \psi &::= \flat\varphi \mid \neg\psi \mid \psi \wedge \psi \mid \psi \vee \psi, \end{aligned}$$

where in the formation rule of  $\wp\psi$  it is ensured that  $\wp \in \text{Qnt}(\text{free}(\psi))$ .

Finally, the simpler  $\text{SL}[\text{1G}]$  formulas are obtained by forcing each goal to be coupled with a quantification prefix:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{X}\varphi \mid \varphi \mathbf{U}\varphi \mid \varphi \mathbf{R}\varphi \mid \wp\flat\varphi,$$

where in the formation rule  $\wp\flat\varphi$  it is ensured that  $\wp \in \text{Qnt}(\text{free}(\flat\varphi))$ .

$\text{SL} \supset \text{SL}[\text{BG}] \supset \text{SL}[\text{1G}]$  denotes the syntactic chain of infinite sets of formulas generated by the respective grammars with the associated constraints on free variables of goals.

Intuitively, in  $\text{SL}[\text{BG}]$  and  $\text{SL}[\text{1G}]$ , we force the writing of formulas to use atomic blocks of quantifications and bindings, where the related free variables are strictly coupled with those that are effectively quantified in the prefix just before the binding. In a nutshell, we can only write formulas by using sentences of the form  $\wp\psi$  belonging to a kind of *prenex normal form* in which the quantifications contained into the *matrix*  $\psi$  only belong to the prefixes  $\wp'$  for some inner subsentence  $\wp'\psi' \in \text{snt}(\wp\psi)$ .

An  $\text{SL}[\text{BG}]$  sentence  $\phi$  is *principal* if it is of the form  $\phi = \wp\psi$ , where  $\psi$  is agent-closed and  $\wp \in \text{Qnt}(\text{free}(\psi))$ . By  $\text{psnt}(\varphi) \subseteq \text{snt}(\varphi)$  we denote the set of all principal subsentences of the formula  $\varphi$ .

In order to practice with the above fragments, let us consider again the sentence  $\varphi_{NE}$  of Example 2.7. It is easy to see that it is not an  $\text{SL}[\text{BG}]$  formula. However, by rearranging quantifications and bindings we can obtain the equivalent formula  $\varphi'_{NE} = \wp \bigwedge_{i=1}^n \flat_i \psi_i \rightarrow \flat \psi_i$ , where  $\wp = \langle\langle \mathbf{x}_1 \rangle\rangle \cdots \langle\langle \mathbf{x}_n \rangle\rangle \llbracket \mathbf{y}_1 \rrbracket \cdots \llbracket \mathbf{y}_n \rrbracket$ ,  $\flat = (\alpha_1, \mathbf{x}_1) \cdots (\alpha_n, \mathbf{x}_n)$ ,  $\flat_i = (\alpha_1, \mathbf{x}_1) \cdots (\alpha_{i-1}, \mathbf{x}_{i-1})(\alpha_i, \mathbf{y}_i)(\alpha_{i+1}, \mathbf{x}_{i+1}) \cdots (\alpha_n, \mathbf{x}_n)$ , and  $\text{free}(\psi_i) = \text{Ag}$ . Now, it is not hard

to see that  $\varphi'_{NE}$ , as well as the equivalent formulations of  $\varphi_{EG}$  and  $\varphi_{AG}$  of Example 2.8 and Example 2.9, respectively, belong to  $\text{SL}[\text{BG}]$  but not to  $\text{SL}[\text{1G}]$ .

In Section 3, we prove the undecidability of the satisfiability problem for  $\text{SL}$ . Now, it is not hard to see that the formula  $\varphi^{\text{dom}}$  used to reduce the domino problem in Theorem 3.10 actually lies in the  $\text{SL}[\text{BG}]$  fragment. Hence, the satisfiability for this logic is undecidable too. On the other hand, later in the paper, we prove that the same problem for  $\text{SL}[\text{1G}]$  is  $2\text{EXPTIME-COMplete}$ , thus not harder than the one for  $\text{ATL}^*$ . We have the following theorem.

**Theorem 4.2.** *The satisfiability problem for  $\text{SL}[\text{BG}]$  is highly undecidable. In particular, it is  $\Sigma_1^1$ -HARD.*

In addition to this, we recall that in [TW12], the authors prove that the satisfiability problem for a logic called  $\text{ATL}^*$  *with strategy context*, introduced in [DLM10] is undecidable also on the class of finite models. It is easy to prove that  $\text{ATL}^*$  with strategy context can be embedded in  $\text{SL}$ . This implies the following.

**Theorem 4.3.** *The satisfiability on finite models problem for  $\text{SL}$  is undecidable.*

**Remark 4.4.** It is important to notice that, since  $\text{ATL}^*$  with strategy context cannot be embedded in  $\text{SL}[\text{BG}]$ , the result provided in [TW12] is not sufficient to prove the undecidability of the satisfiability problem for  $\text{SL}[\text{BG}]$ . Moreover, the result shown here is stronger, as it proves highly undecidability for  $\text{SL}[\text{BG}]$ . On the contrary, in [MMPV14], we introduce a fragment called *"Nested-Goal Strategy Logic"* ( $\text{SL}[\text{NG}]$ , for short), that strictly subsumes  $\text{SL}[\text{BG}]$  and in which  $\text{ATL}^*$  with strategy context can be embedded. This fragment, then, is undecidable on finite models and highly undecidable on infinite models. In addition to this, notice that in [LM13] the authors show that the decidability problem for  $\text{SL}$  under turn-based CGSs is decidable. This means that the undecidability under concurrent CGSs is strict.

**4.2. Behavioral semantics.** We now recall the fundamental property of *behavioral semantics* enjoyed by  $\text{SL}[\text{1G}]$ . All concepts and results have been already introduced and fully investigated in [MMPV14]. We report them here for the sake of completeness.

We first need to describe the concept of Skolem dependence function (**Sdf**, for short) and show how any quantification prefix contained into an  $\text{SL}$  formula can be represented by an adequate choice of a **Sdf** over strategies. The main idea here is inspired by the technique proposed by Skolem for the first-order logic in order to eliminate all existential quantifications over variables, by substituting them with second order existential quantifications over functions, whose choice is uniform *w.r.t.* the universal variables.

We first introduce some notation regarding the quantification prefixes. Let  $\varphi \in \text{Qnt}(\text{V})$  be a quantification prefix over a set  $\text{V} \subseteq \text{Vr}$  of variables. By  $\langle\langle\varphi\rangle\rangle \triangleq \{x \in \text{V}(\varphi) : \exists i \in [0, |\varphi|[, (\varphi)_i = \langle\langle x \rangle\rangle\}$  and  $\llbracket\varphi\rrbracket \triangleq \text{V}(\varphi) \setminus \langle\langle\varphi\rangle\rangle$  we denote the sets of *existential* and *universal variables* quantified in  $\varphi$ , respectively. For two variables  $x, y \in \text{V}(\varphi)$ , we say that  $x$  *precedes*  $y$  in  $\varphi$ , in symbols  $x <_{\varphi} y$ , if  $x$  occurs before  $y$  in  $\varphi$ , *i.e.*, there are two indexes  $i, j \in [0, |\varphi|[,$  with  $i < j$ , such that  $(\varphi)_i \in \{\langle\langle x \rangle\rangle, \llbracket x \rrbracket\}$  and  $(\varphi)_j \in \{\langle\langle y \rangle\rangle, \llbracket y \rrbracket\}$ . Moreover, we say that  $y$  is *functional dependent* on  $x$ , in symbols  $x \rightsquigarrow_{\varphi} y$ , if  $x \in \llbracket\varphi\rrbracket$ ,  $y \in \langle\langle\varphi\rangle\rangle$ , and  $x <_{\varphi} y$ , *i.e.*,  $y$  is existentially quantified after that  $x$  is universally quantified, so, there may be a dependence between a value chosen by  $x$  and that chosen by  $y$ . This definition induces the

set  $\text{Dep}(\varphi) \triangleq \{(x, y) \in V(\varphi) \times V(\varphi) : x \rightsquigarrow_{\varphi} y\}$  of *dependence pairs* and its derived version  $\text{Dep}(\varphi, y) \triangleq \{x \in V(\varphi) : x \rightsquigarrow_{\varphi} y\}$  containing all variables from which  $y$  depends. Finally, we use  $\overline{\varphi} \in \text{Qnt}(V(\varphi))$  to indicate the quantification derived from  $\varphi$  by *dualizing* each quantifier contained in it, *i.e.*, for all indexes  $i \in [0, |\varphi|]$ , it holds that  $(\overline{\varphi})_i = \langle\langle x \rangle\rangle$  iff  $(\varphi)_i = \llbracket x \rrbracket$ , with  $x \in V(\varphi)$ . It is evident that  $\langle\langle \overline{\varphi} \rangle\rangle = \llbracket \varphi \rrbracket$  and  $\llbracket \overline{\varphi} \rrbracket = \langle\langle \varphi \rangle\rangle$ . As an example, let  $\varphi = \llbracket \mathbf{x} \rrbracket \langle\langle \mathbf{y} \rangle\rangle \langle\langle \mathbf{z} \rangle\rangle \llbracket \mathbf{w} \rrbracket \langle\langle \mathbf{v} \rangle\rangle$ . Then, we have  $\langle\langle \varphi \rangle\rangle = \{\mathbf{y}, \mathbf{z}, \mathbf{v}\}$ ,  $\llbracket \varphi \rrbracket = \{\mathbf{x}, \mathbf{w}\}$ ,  $\text{Dep}(\varphi, \mathbf{x}) = \text{Dep}(\varphi, \mathbf{w}) = \emptyset$ ,  $\text{Dep}(\varphi, \mathbf{y}) = \text{Dep}(\varphi, \mathbf{z}) = \{\mathbf{x}\}$ ,  $\text{Dep}(\varphi, \mathbf{v}) = \{\mathbf{x}, \mathbf{w}\}$ , and  $\overline{\varphi} = \langle\langle \mathbf{x} \rangle\rangle \llbracket \mathbf{y} \rrbracket \llbracket \mathbf{z} \rrbracket \langle\langle \mathbf{w} \rangle\rangle \llbracket \mathbf{v} \rrbracket$ .

Finally, we define the notion of *valuation* of variables over a generic set  $D$ , called *domain*, *i.e.*, a partial function  $\mathbf{v} : \text{Vr} \rightarrow D$  mapping every variable in its domain to an element in  $D$ . By  $\text{Val}_D(V) \triangleq V \rightarrow D$  we denote the set of all valuation functions over  $D$  defined on  $V \subseteq \text{Vr}$ .

At this point, we give a general high-level semantics for the quantification prefixes by means of the following definition of *Skolem dependence function*.

**Definition 4.5** (Skolem Dependence Function). Let  $\varphi \in \text{Qnt}(V)$  be a quantification prefix over a set  $V \subseteq \text{Vr}$  of variables, and  $D$  a set. Then, a *Skolem dependence function* for  $\varphi$  over  $D$  is a function  $\theta : \text{Val}_D(\llbracket \varphi \rrbracket) \rightarrow \text{Val}_D(V)$  satisfying the following two properties:

- (1)  $\theta(\mathbf{v}) \upharpoonright_{\llbracket \varphi \rrbracket} = \mathbf{v}$ , for all  $\mathbf{v} \in \text{Val}_D(\llbracket \varphi \rrbracket)$ ; <sup>9</sup>
- (2)  $\theta(\mathbf{v}_1)(x) = \theta(\mathbf{v}_2)(x)$ , for all  $\mathbf{v}_1, \mathbf{v}_2 \in \text{Val}_D(\llbracket \varphi \rrbracket)$  and  $x \in \langle\langle \varphi \rangle\rangle$  such that  $\mathbf{v}_1 \upharpoonright_{\text{Dep}(\varphi, x)} = \mathbf{v}_2 \upharpoonright_{\text{Dep}(\varphi, x)}$ .

$\text{SF}_D(\varphi)$  denotes the set of all **Sdfs** for  $\varphi$  over  $D$ .

Intuitively, Item 1 asserts that  $\theta$  assumes the same values of its argument *w.r.t.* the universal variables in  $\varphi$ , while Item 2 ensures that the value of  $\theta$  *w.r.t.* an existential variable  $x$  in  $\varphi$  does not depend on variables not in  $\text{Dep}(\varphi, x)$ . To get a better insight, note that a **Sdf**  $\theta$  for  $\varphi$  can be considered as a set of classical *Skolem functions* that, given a value for each variable in  $\llbracket \varphi \rrbracket$  returns a possible value for all variables in  $\langle\langle \varphi \rangle\rangle$ , in a way that is consistent *w.r.t.* the order of quantifications. Observe that, each  $\theta \in \text{SF}_D(\varphi)$  is injective, so,  $|\text{rng}(\theta)| = |\text{dom}(\theta)| = |D|^{\llbracket \varphi \rrbracket}$ . Moreover,  $|\text{SF}_D(\varphi)| = \prod_{x \in \langle\langle \varphi \rangle\rangle} |D|^{|D|^{\text{Dep}(\varphi, x)}}$ . As an example, let  $D = \{0, 1\}$  and  $\varphi = \llbracket \mathbf{x} \rrbracket \langle\langle \mathbf{y} \rangle\rangle \llbracket \mathbf{z} \rrbracket \in \text{Qnt}(V)$  be a quantification prefix over  $V = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ . Then, we have that  $|\text{SF}_D(\varphi)| = 4$  and  $|\text{SF}_D(\overline{\varphi})| = 8$ . Moreover, the **Sdfs**  $\theta^i \in \text{SF}_D(\varphi)$  with  $i \in [0, 3]$  and  $\overline{\theta}^i \in \text{SF}_D(\overline{\varphi})$  with  $i \in [0, 7]$ , for a particular fixed order, are such that  $\theta^0(\mathbf{v})(\mathbf{y}) = 0$ ,  $\theta^1(\mathbf{v})(\mathbf{y}) = \mathbf{v}(\mathbf{x})$ ,  $\theta^2(\mathbf{v})(\mathbf{y}) = 1 - \mathbf{v}(\mathbf{x})$ , and  $\theta^3(\mathbf{v})(\mathbf{y}) = 1$ , for all  $\mathbf{v} \in \text{Val}_D(\llbracket \varphi \rrbracket)$ , and  $\overline{\theta}^i(\overline{\mathbf{v}})(\mathbf{x}) = 0$  with  $i \in [0, 3]$ ,  $\overline{\theta}^i(\overline{\mathbf{v}})(\mathbf{x}) = 1$  with  $i \in [4, 7]$ ,  $\overline{\theta}^0(\overline{\mathbf{v}})(\mathbf{z}) = \overline{\theta}^4(\overline{\mathbf{v}})(\mathbf{z}) = 0$ ,  $\overline{\theta}^1(\overline{\mathbf{v}})(\mathbf{z}) = \overline{\theta}^5(\overline{\mathbf{v}})(\mathbf{z}) = \overline{\mathbf{v}}(\mathbf{y})$ ,  $\overline{\theta}^2(\overline{\mathbf{v}})(\mathbf{z}) = \overline{\theta}^6(\overline{\mathbf{v}})(\mathbf{z}) = 1 - \overline{\mathbf{v}}(\mathbf{y})$ , and  $\overline{\theta}^3(\overline{\mathbf{v}})(\mathbf{z}) = \overline{\theta}^7(\overline{\mathbf{v}})(\mathbf{z}) = 1$ , for all  $\overline{\mathbf{v}} \in \text{Val}_D(\llbracket \overline{\varphi} \rrbracket)$ .

We now report the following fundamental theorem that describes how to eliminate the strategy quantifications of an SL formula via a choice of a suitable **Sdf** over strategies [MMPV14]. This procedure can be seen as the equivalent of the *Skolemization* procedure in first-order logic (see [Hod93], for more details).

**Theorem 4.6** (SL Strategy Quantification). *Let  $\mathcal{G}$  be a CGS and  $\varphi = \varphi\psi$  an SL sentence, where  $\psi$  is agent-closed and  $\varphi \in \text{Qnt}(\text{free}(\psi))$ . Then,  $\mathcal{G} \models \varphi$  iff there exists a **Sdf**  $\theta \in \text{SF}_{\text{Str}}(\varphi)$  such that  $\mathcal{G}, \theta(\chi), s_0 \models \psi$ , for all  $\chi \in \text{Asg}(\llbracket \varphi \rrbracket, s_0)$ .*

<sup>9</sup>By  $\mathbf{g}|_Z : (X \cap Z) \rightarrow Y$  we denote the *restriction* of a function  $\mathbf{g} : X \rightarrow Y$  to the elements in the set  $Z$ .

We now restrict our attention to a particular subclass of **Sdfs** defined on strategies called *behavioral Skolem dependence functions*. Intuitively, an **Sdf** behavioral on strategies can be split into an infinite set of **Sdfs** over actions, one per each track in the domains of strategies. As next definition clarifies, not all the **Sdfs** are behavioral. This means that the announced simplification applies only under certain conditions.

**Definition 4.7** (Adjoint Functions). Let  $\theta : \text{Val}_{\text{Str}}(\llbracket\wp\rrbracket) \rightarrow \text{Val}_{\text{Str}}(\text{Vr})$  be an **Sdf** on strategies and let  $\tilde{\theta} : \text{Trk} \rightarrow (\text{Val}_{\text{Ac}}(\llbracket\wp\rrbracket) \rightarrow \text{Val}_{\text{Ac}}(\text{Vr}))$  be a function mapping every track into a **Sdf** on actions. We say that  $\tilde{\theta}$  is the *adjoint* of  $\theta$  if  $\tilde{\theta}(\rho)(\widehat{\chi}(\rho))(x) = \theta(\chi)(x)(\rho)$ , for all  $\chi \in \text{Asg}_{\text{Str}}(\llbracket\wp\rrbracket)$ ,  $x \in \text{Vr}$ , and  $\rho \in \text{Trk}$ <sup>10</sup>.

Intuitively,  $\tilde{\theta}$  is the adjoint of  $\theta$  if the dependence from tracks in  $\text{Trk}$  in both domain and codomain of the latter function can be extracted and put as a common factor of the former function. This implies also that, for every pair of functions  $\chi_1, \chi_2 \in \text{Asg}_{\text{Str}}(\llbracket\wp\rrbracket)$  such that  $\widehat{\chi}_1(\rho) = \widehat{\chi}_2(\rho)$  for some  $\rho \in \text{Trk}$ , it holds that  $\theta(\chi_1)(x)(\rho) = \theta(\chi_2)(x)(\rho)$ , for all variables  $x \in \text{Vr}$ . It is immediate to observe that if a function has an adjoint then that adjoint is unique. At the same way, from an adjoint function it is possible to determine the original function without any ambiguity. Thus, it is established a one-to-one correspondence between functions admitting an adjoint and the adjoints themselves.

We have the following definition.

**Definition 4.8.** An **Sdf** is called *behavioral* if it admits the adjoint function. Moreover, by  $\text{BSF}_{\text{Str}}(\wp)$  we denote the set of behavioral **Sdfs** for  $\wp$  over the set of strategies  $\text{Str}$

It is proved in [MMPV14] that a necessary and sufficient condition for a function  $\tilde{\theta}$  to be an adjoint of a certain **Sdf**  $\theta \in \text{SF}_{\text{Str}}(\wp)$  is that  $\tilde{\theta}(\rho)$  is in  $\text{SF}_{\text{Ac}}(\wp)$ , for all  $\rho \in \text{Trk}$ .

Unfortunately, not every **Sdf** has an adjoint function. An easy way to prove this, it is to the number of **Sdfs** and adjoints. Indeed, we have that

$$|\text{SF}_{\text{Str}}(\wp)| = \prod_{x \in \langle\langle\wp\rangle\rangle} |\text{Ac}|^{|\text{Trk}| \cdot |\text{Ac}|^{|\text{Trk}| \cdot |\text{Dep}(\wp, x)|}},$$

which is doubly exponential in the set  $\text{Trk}$  of tracks, while

$$|\text{Trk} \rightarrow \text{SF}_{\text{Ac}}(\wp)| = \prod_{x \in \langle\langle\wp\rangle\rangle} |\text{Ac}|^{|\text{Trk}| \cdot |\text{Ac}|^{|\text{Dep}(\wp, x)|}},$$

which is only singly exponential in the same set  $\text{Trk}$ .

**Definition 4.9** (SL[BG] Behavioral Semantics [MMPV14]). Let  $\mathcal{G}$  be a CGS,  $s \in \text{St}$  one of its states, and  $\wp\psi$  an SL[BG] formula, where  $\psi$  is agent-closed and  $\wp \in \text{Qnt}(\text{free}(\psi))$ . Then  $\mathcal{G}, s \models_{\text{B}} \wp\psi$  if there exists a behavioral **Sdf**  $\theta \in \text{BSF}_{\text{Str}}(\wp)$  for  $\wp$  over  $\text{Str}$  such that  $\mathcal{G}, \theta(\chi), s \models_{\text{B}} \psi$ , for all  $\chi \in \text{Asg}(\llbracket\wp\rrbracket, s)$ .

Clearly, from the previous definition and Theorem 4.6, we have that  $\mathcal{G} \models_{\text{B}} \wp$  implies  $\mathcal{G} \models \wp$ . In [MMPV14] it has been shown that the converse may not hold in general, *i.e.*, there exists a CGS  $\mathcal{G}$  and a SL[BG] formula  $\wp$  such that  $\mathcal{G} \models \wp$  but  $\mathcal{G} \not\models_{\text{B}} \wp$ . However, as a fundamental result for the SL[1G] fragment, in [MMPV14] it has also been proved that the behavioral semantics is equivalent to the classic one. This fact is derived by means of a

<sup>10</sup>By  $\widehat{g} : Y \rightarrow X \rightarrow Z$  we denote the operation of *flipping* of a function  $g : X \rightarrow Y \rightarrow Z$ .

reduction from the verification problem of a  $\text{SL}[1\text{G}]$  sentence against a CGS to the winning problem of a Borelian two-player game.

**Theorem 4.10** (SL[1G] Behavioral [MMPV14]). *Let  $\mathcal{G}$  be a CGS and  $\varphi$  an SL[1G] sentence. Then,  $\mathcal{G} \models \varphi$  iff  $\mathcal{G} \models_{\text{B}} \varphi$ .*

It is important to note that the behavioral property of SL[1G] is fundamental in proving many positive properties of the logic, as the bounded model property, which lead to a decidable procedure for the satisfiability problem, as we show later in the paper.

**4.3. Tree-model property.** The satisfiability procedure we propose later in the paper is based on the use of alternating tree automata. Consequently, we need to establish a kind of *tree model property*, which is based on a special sub-class of CGSs, namely, the *concurrent game-trees* (CGT, for short), whose structure of the underlying graph is a tree.

**Definition 4.11** (Concurrent Game Trees). A *concurrent game tree* (CGT, for short) is a CGS  $\mathcal{T} \triangleq \langle \text{AP}, \text{Ag}, \text{Ac}, \text{St}, \text{tr}, \text{ap}, s_0 \rangle$ , where (i)  $\text{St} \subseteq \text{Dir}^*$  is a Dir-tree for a given set  $\text{Dir}$  of directions and (ii) if  $t \cdot e \in \text{St}$  then there is a decision  $\delta \in \text{Dc}$  such that  $\text{tr}(t, \delta) = t \cdot e$ , for all  $t \in \text{St}$  and  $e \in \text{Dir}$ . Furthermore,  $\mathcal{T}$  is a *decision tree* (DT, for short) if (i)  $\text{St} = \text{Dc}^*$  and (ii) if  $t \cdot \delta \in \text{St}$  then  $\text{tr}(t, \delta) = t \cdot \delta$ , for all  $t \in \text{St}$  and  $\delta \in \text{Dc}$ .

Intuitively, CGTs are CGSs having a transition relation with a tree shape and DTs have, in addition, the states that uniquely determine the history of the computation leading to them. Observe that, for each non trivial track  $\rho$  (resp., path  $\pi$ ) of a CGT, there exists a unique finite (resp., infinite) sequence of decisions  $\delta_0 \dots \delta_{|\rho|-2} \in \text{Dc}^*$  (resp.,  $\delta_0 \cdot \delta_1 \dots \in \text{Dc}^\omega$ ) such that  $\rho_{(i+1)} = \text{tr}(\rho_{(i)}, \delta_i)$  (resp.,  $\pi_{(i+1)} = \text{tr}(\pi_{(i)}, \delta_i)$ ), for all  $i \in [0, |\rho| - 1[$  (resp.,  $i \in \mathbb{N}$ ).

We now define a generalization for CGSs of the classic concept of *unwinding* of labeled transition systems, namely the *decision-unwinding* (see Figure 5, for an example), that allows to show that SL[1G] enjoys the decision-tree model property.

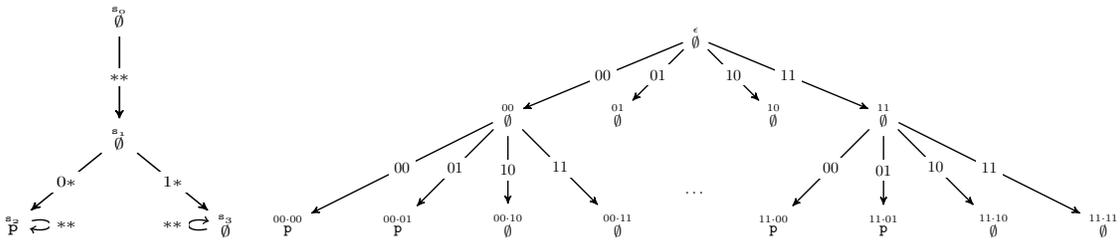


Figure 5: A CGS and part of its decision-unwinding.

**Definition 4.12** (Decision-Unwinding). Let  $\mathcal{G} = \langle \text{AP}, \text{Ag}, \text{Ac}, \text{St}, \text{tr}, \text{ap}, s_0 \rangle$  be a CGS. Then, the *decision-unwinding* of  $\mathcal{G}$  is the DT  $\mathcal{G}_{DU} \triangleq \langle \text{AP}, \text{Ag}, \text{Ac}, \text{Dc}^*, \text{ap}', \text{tr}', \varepsilon \rangle$  for which there is a surjective function  $\text{unw} : \text{Dc}^* \rightarrow \text{St}$  such that (i)  $\text{unw}(\varepsilon) = s_0$ , (ii)  $\text{unw}(\text{tr}'(t, \delta)) = \text{tr}(\text{unw}(t), \delta)$ , and (iii)  $\text{ap}'(t) = \text{ap}(\text{unw}(t))$ , for all  $t \in \text{Dc}^*$  and  $\delta \in \text{Dc}$ .

Observe that, due to its construction, each CGS  $\mathcal{G}$  has a unique associated decision-unwinding  $\mathcal{G}_{DU}$ .

We are now able to prove that SL[1G] satisfies the decision-tree model property.

**Theorem 4.13** (SL[1G] Decision-Tree Model Property). *Let  $\varphi$  be a satisfiable SL[1G] sentence. Then, there exists a DT  $\mathcal{T}$  such that  $\mathcal{T} \models \varphi$ .*

*Proof.* The proof proceeds by structural induction on the sentence SL[1G]. For the Boolean combination of principal sentences, the induction is trivial. For the case of a principal sentence  $\varphi$  of the form  $\wp b \psi$ , by Theorem 4.10, we derive that there exists a behavioral **Sdf**  $\theta \in \text{BSF}_{\text{Str}_{\mathcal{G}}}(\wp)$  such that  $\mathcal{G} \models_{\theta} \wp b \psi$ . Furthermore, there exists the adjoint function tracking  $\tilde{\theta}$  into  $\theta$ . Now, consider the decision unwinding  $\mathcal{T} = \mathcal{G}_{DU}$  of  $\mathcal{G}$  and the lifting  $\Gamma : \text{Trk}_{\mathcal{T}} \rightarrow \text{Trk}_{\mathcal{G}}$  of the unwinding function  $\text{unw}$  such that  $\Gamma(\rho) = \text{unw}(\rho_0) \cdot \dots \cdot \text{unw}(\rho_{|\rho|-1})$ , for all  $\rho \in \text{Trk}_{\mathcal{T}}$ . At this point, consider the function  $\tilde{\theta}'$  such that  $\tilde{\theta}'(\rho') \triangleq \tilde{\theta}(\Gamma(\rho'))$ , for all  $\rho' \in \text{Trk}_{\mathcal{T}}$ . Clearly, since  $\tilde{\theta}$  is a **Sdf** over actions, so  $\tilde{\theta}'$  is as well. Then, consider the **Sdf**  $\theta'$  for which the function  $\tilde{\theta}'$  is its adjoint. By induction on the nesting of principal subsentences in  $\varphi = \wp b \psi$ , we now prove that  $\mathcal{T} \models_{\theta'} \wp b \psi$ . As base case, *i.e.*, when  $\psi$  is an LTL formula, consider an assignment  $\chi' \in \text{Asg}_{\mathcal{T}}(\llbracket \wp \rrbracket, \varepsilon)$  and the induced play  $\pi' = \text{play}(\theta'(\chi') \circ b, \varepsilon)$  over  $\mathcal{T}$ . Moreover, consider an assignment  $\chi \in \text{Asg}_{\mathcal{G}}(\llbracket \wp \rrbracket, s_0)$  such that, for all placeholders  $l \in \text{dom}(\chi)$  and tracks  $\rho' \in \text{Trk}_{\mathcal{T}}$ , it holds that  $\chi(l)(\Gamma(\rho')) = \chi'(l)(\rho')$ . From the satisfiability of  $\varphi$  on  $\mathcal{G}$ , we derive that  $\pi \models \psi$ , where  $\pi = \text{play}(\theta(\chi) \circ b, s_0)$ . Indeed, assume for a while that  $(\pi)_{\leq k} = \Gamma((\pi')_{\leq k})$ . Then, from the definition of the labeling in the decision-tree unwinding, it easily follows that  $\text{ap}((\pi)_k) = \text{ap}'((\pi')_k)$ , and, so we derive  $\pi' \models \psi$ , from  $\pi \models \psi$ . Consequently, this holds for all  $\chi' \in \text{Asg}_{\text{Str}_{\mathcal{T}}}(\llbracket \wp \rrbracket)$  and, so, we have that  $\mathcal{T} \models_{\theta'} \wp b \psi$ . The inductive case, *i.e.*, when  $\psi$  contains some subsentence, easily follows by considering the principal subsentences as fresh atomic propositions.

It remains to prove, by induction on  $k$ , that  $(\pi)_{\leq k} = \Gamma((\pi')_{\leq k})$ , for all  $k \in \mathbb{N}$ . As base case, we have that  $(\pi)_0 = s_0 = \Gamma(\varepsilon) = (\pi')_0$ . As inductive case, assume that  $(\pi)_{\leq k} = \Gamma((\pi')_{\leq k})$ . Then, in particular, we have that  $(\pi)_k = \Gamma((\pi')_k) = \text{unw}((\pi')_k)$ . By definition of play, it holds that  $(\pi)_{k+1} = \text{tr}((\pi)_k, (\tilde{\theta}((\pi)_{\leq k}))(\hat{\chi})) \circ b$ , which is, by inductive hypothesis, equal to  $\text{tr}(\text{unw}((\pi')_k), (\tilde{\theta}(\Gamma((\pi)_{\leq k}))) (\hat{\chi})) \circ b$ . Now, by the definition of  $\tilde{\theta}'$  and  $\chi$ , we obtain  $\text{tr}(\text{unw}((\pi')_k), (\tilde{\theta}(\Gamma((\pi)_{\leq k}))) (\hat{\chi})) \circ b = \text{tr}(\Gamma((\pi')_k), (\tilde{\theta}'((\pi')_{\leq k})) (\hat{\chi}')) \circ b$ . Finally, by the definition of  $\text{unw}$  and  $\Gamma$ , we have that  $\text{tr}(\Gamma((\pi')_k), (\tilde{\theta}'((\pi')_{\leq k})) (\hat{\chi}')) \circ b = \Gamma((\pi')_{k+1})$ .  $\square$

## 5. DECIDABILITY OF SL[1G]

In this section, we finally provide a 2EXPTIME-COMPLETE procedure for the SL[1G] satisfiability problem. Before doing this, we have to prove the *bounded-tree model property*, which results to be crucial for the automata-theoretic approach later described.

**5.1. Bounded model property.** In order to prove the bounded-tree model property for SL[1G], we first need to introduce the new concept of *disjoint satisfiability*, which concerns the verification of different instances of the same subsentence of the original specification. Intuitively, it asserts that either these instances can be checked on disjoint subtrees of the tree model or, if two instances use part of the same subtree, they are forced to use the same dependence map as well. This notion is a reformulation of the notion of explicit model introduced for ATL\* in [Sch08]. This intrinsic characteristic of SL[1G] is fundamental for the building of a unique automaton that checks the truth of all subsentences, by simply

merging their respective automata, without using a projection operation to eliminate their own alphabets, which otherwise may be in conflict. In this way, we are also able to avoid an exponential blow-up. A deeper discussion on this point is reported later in the paper.

**Definition 5.1** (Disjoint Satisfiability). Let  $\mathcal{T}$  be a DT and  $\varphi = \wp \flat \psi$  be a SL[1G] principal sentence. Moreover, let  $S \triangleq \{s \in \text{St}_{\mathcal{T}} : \mathcal{T}, s \models \varphi\}$ . Then,  $\mathcal{T}$  satisfies  $\varphi$  *disjointly* over  $S$  if there exist two functions  $\text{head} : S \rightarrow \text{SF}_{\text{Ac}}(\wp)$  and  $\text{body} : \text{Trk}(\varepsilon) \rightarrow \text{SF}_{\text{Ac}}(\wp)$  such that, for all  $s \in S$  and  $\chi \in \text{Asg}_{\text{Str}}(\llbracket \wp \rrbracket)$  it holds that  $\mathcal{T}, \theta(\chi), s \models \flat \psi$ , where the behavioral Sdf  $\theta \in \text{BSF}_{\text{Str}}$  is defined, by means of its adjoint, as follows:

- (i)  $\tilde{\theta}(s) \triangleq \text{head}(s)$ ;
- (ii)  $\tilde{\theta}(\rho) \triangleq \text{body}(\rho' \cdot \rho)$ , for all  $\rho \in \text{Trk}(s)$  with  $|\rho| > 1$ , where  $\rho' \in \text{Trk}(\varepsilon)$  is the unique track such that  $\rho' \cdot \rho \in \text{Trk}(\varepsilon)$  <sup>11</sup>.

The disjoint satisfiability holds for all SL[1G] formulas. To prove this fact, we first introduce the preliminary definition of *twin decision-tree*. Intuitively, in such a kind of tree, each action is flanked by a twin one having the same purpose of the original. This allows to satisfy two sentences requiring the same actions in a given state in two different branches of the tree itself, which is what the disjoint satisfiability precisely requires.

**Definition 5.2** (Twin Decision Tree). Let  $\mathcal{T} = \langle \text{AP}, \text{Ag}, \text{Ac}, \text{St}, \text{tr}, \text{ap}, \varepsilon \rangle$  be a DT. Then, the *twin decision tree* of  $\mathcal{T}$  is the DT  $\mathcal{T}' \triangleq \langle \text{AP}, \text{Ag}, \text{Ac}', \text{St}', \text{tr}', \text{ap}', \varepsilon' \rangle$  with  $\text{Ac}' = \text{Ac} \times \{\text{new}, \text{cont}\}$  and  $\varepsilon' = (\varepsilon, \text{new})$ . The labeling and the transition function are defined by means of a set of projection functions introduced below:

- the function  $\text{prj}_{\text{Ac}} : \text{Ac}' \rightarrow \text{Ac}$  returns the first component of the action in  $\text{Ac}'$ , *i.e.*,  $\text{prj}((c, \iota)) = c$ , for all  $(c, \iota) \in \text{Ac}'$ ;
- the function  $\text{prj}_{\text{Dc}} : \text{Dc}' \rightarrow \text{Dc}$  projects out the flags on all the actions in the decision, returning a corresponding decision in  $\mathcal{T}$ , *i.e.*,  $\text{prj}_{\text{Dc}}(\delta')(a) = \text{prj}_{\text{Ac}}(\delta'(a))$ , for all  $\delta' \in \text{Dc}'$  and  $a \in \text{Ag}$ ;
- the function  $\text{prj}_{\text{St}} : \text{St}' \rightarrow \text{St}$  returns the corresponding state in  $\mathcal{T}$ , according to the projection made on the decisions, *i.e.*,  $\text{prj}_{\text{St}}(\varepsilon') = \varepsilon$  and  $\text{prj}_{\text{St}}(s' \cdot \delta') = \text{prj}_{\text{St}}(s') \cdot \text{prj}_{\text{Dc}}(\delta')$ , for all  $s' \in \text{St}'$  and  $\delta' \in \text{Dc}'$ ;
- analogously, the function  $\text{prj}_{\text{Trk}} : \text{Trk}' \rightarrow \text{Trk}$ , returns the concatenation of the projected states, *i.e.*,  $\text{prj}_{\text{Trk}}(s') = \text{prj}_{\text{St}}(s')$ , for all  $s' \in \text{St}'$ , and  $\text{prj}_{\text{Trk}}(\rho' \cdot s') = \text{prj}_{\text{Trk}}(\rho') \cdot \text{prj}_{\text{St}}(s')$ , for all  $\rho' \in \text{Trk}'$  and  $s' \in \text{St}'$ .

Then,  $\text{ap}'(s') \triangleq \text{ap}(\text{prj}_{\text{St}}(s'))$ .

Observe that  $\text{prj}_{\text{St}}(\text{tr}'(s', \delta')) = \text{tr}(\text{prj}_{\text{St}}(s'), \text{prj}_{\text{Dc}}(\delta'))$ , for all  $s' \in \text{St}'$  and  $\delta' \in \text{Dc}'$ . We can now prove the disjoint satisfiability property for SL[1G].

**Theorem 5.3** (Disjoint Satisfiability). *Let  $\varphi = \wp \flat \psi$  be an SL[1G] principal sentence and  $\mathcal{T} = \langle \text{AP}, \text{Ag}, \text{Ac}, \text{St}, \text{tr}, \text{ap}, \varepsilon \rangle$  a DT. Moreover, let  $S \triangleq \{s \in \text{St} : \mathcal{T}, s \models \varphi\}$ . Then the twin decision tree  $\mathcal{T}'$  of  $\mathcal{T}$  disjointly satisfies  $\varphi$  over  $S' \triangleq \{s' \in \text{St}' : \text{prj}(s') \in S\}$ .*

*Proof idea.* Starting from the fact that  $\mathcal{T}, s \models \varphi$ , for all  $s \in S$ , by means of Theorem 4.10, we derive the existence of a behavioral Sdf  $\theta_s$ . Such a  $\theta_s$  is used to define a behavioral Sdf  $\theta_{s'}$  in  $\mathcal{T}'$  in which the existential agents suitably select either *new* or *cont* as second component, in order to guarantee the satisfaction of different instances over different branches of the twin decision tree. Indeed, it allows to properly define the two functions *head* and *body*

<sup>11</sup>Existence and uniqueness of  $\rho'$  is guaranteed by the fact that  $\mathcal{T}$  is a DT.

and, consequently, the behavioral  $\mathbf{Sdf}$   $\theta'$  for which we finally prove that  $\mathcal{T}', s' \models \varphi$ , for all  $s' \in S'$ . Since  $\theta'$  has been built from the head and body functions, the disjoint satisfiability is immediately derived.  $\square$

*Proof.* Let  $s \in S$  be one of the states on which  $\varphi$  is satisfied. Since  $\mathcal{T}, s \models \varphi$ , by Theorem 4.10, we have that there exists  $\theta_s \in \mathbf{BSF}_{\text{Str}}(\varphi)$  such that  $\mathcal{T}, \theta_s(\chi), s \models \mathfrak{b}\psi$ , for all states assignments  $\chi \in \text{Asg}_{\text{Str}_{\mathcal{T}}}(\llbracket \varphi \rrbracket)$ . Then, consider the adjoint function  $\tilde{\theta}'_s : \text{Trk}' \rightarrow \mathbf{SF}_{\text{Ac}'}(\varphi)$  defined, for all states  $s' \in \text{St}'$ , decisions  $\delta' \in \text{Dc}'$ , and tracks  $\rho' \in \text{Trk}'$  as follows:

- $\tilde{\theta}'_s(s')(\delta')(x) = (\tilde{\theta}_s(\text{prj}_{\text{St}}(s')))(\text{prj}_{\text{Dc}}(\delta'))(x, \text{new})$ , if  $\text{prj}_{\text{St}}(s') = s$ ;
- $\tilde{\theta}'_s(\rho')(\delta')(x) = (\tilde{\theta}_s(\text{prj}_{\text{Trk}}(\rho')))(\text{prj}_{\text{Dc}}(\delta'))(x, \text{cont})$ , otherwise.

At this point, we assume the function  $\text{head} : \text{St}' \rightarrow \mathbf{SF}_{\text{Ac}'}(\varphi)$  to be defined as follows:  $\text{head}(s') \triangleq \tilde{\theta}'_s(s')$ . Moreover, we set the function  $\text{body} : \text{Trk}'(\varepsilon) \rightarrow \mathbf{SF}_{\text{Ac}'}(\varphi)$  in such a way that it agrees with  $\tilde{\theta}'_s$  on all tracks  $\rho' = s'_0 \dots s'_n$  for which there is an index  $i \in \{0, \dots, n\}$  such that, for all agents  $a \in \text{Ag}$  with  $\mathfrak{b}(a) \in \llbracket \varphi \rrbracket$ , it holds that:

- $\text{lst}((\rho')_i)(a) = (c_a, \text{new})$ , for some  $c_a \in \text{Ac}$ , and
- $\text{lst}((\rho')_j)(a) = (c_a, \text{cont})$ , for all  $j \in \{i + 1, \dots, n\}$  and for some  $c_a \in \text{Ac}$ .

Note that the tracks of this form are such that the players bound to an existentially quantified variable have selected an action flagged by *new* on the  $i$ -th step of the game and then keep playing with the *cont* flag. Intuitively, they are starting the verification of a subsentence right in the  $i$ -th state of the track, by keeping it separated from the verification of the other subsentences, which are addressed with the *cont* flag.

For all the other tracks  $\rho'$ , instead, the value of  $\text{body}(\rho')$  may be arbitrary.

Now, consider the behavioral  $\mathbf{Sdf}$   $\theta' \in \mathbf{BSF}_{\text{Str}}(\varphi)$  defined by means of the functions  $\text{head}$  and  $\text{body}$  as prescribed by Definition 5.1. It remains to prove that  $\mathcal{T}', s' \models_{\theta'} \varphi \mathfrak{b}\psi$ , for all  $s' \in S'$ . We proceed by induction on the nesting of the principal subsentences of  $\varphi$ . As base case, assume that such nesting is 0. This means that  $\psi$  is an LTL formula. Now, let  $\chi' \in \text{Asg}_{\text{Str}'}(\llbracket \varphi \rrbracket)$ . By construction, it is not hard to see that there exists an assignment  $\chi \in \text{Asg}_{\text{Str}}(\llbracket \varphi \rrbracket)$  for which the play  $\pi' \triangleq \text{play}'(\theta'(\chi') \circ \mathfrak{b}, s')$  satisfies the equality  $\text{prj}_{\text{Pth}}(\pi') = \text{play}(\theta(\chi) \circ \mathfrak{b}, \text{prj}(s')) = \pi$ <sup>12</sup>. Thus, since  $\mathcal{T}, s \models_{\theta} \varphi \mathfrak{b}\psi$ , we have that  $\pi \models \psi$ . Moreover, it holds that  $\text{ap}'((\pi')_i) = \text{ap}((\pi)_i)$ , for all  $i \in \mathbb{N}$ , which implies that  $\pi' \models \psi$ . Consequently, we can conclude that  $\mathcal{T}', s' \models_{\theta'} \varphi \mathfrak{b}\psi$ . The inductive case, easily follows by considering the inner principal subsentences as fresh atomic propositions.  $\square$

We now have all tools to prove the bounded model property of  $\text{SL}[1G]$ .

**Theorem 5.4** (Bounded Model Property of  $\text{SL}[1G]$ ). *Let  $\varphi$  be a  $\text{SL}[1G]$  sentence and  $\mathcal{T}$  be a DT such that  $\mathcal{T} \models \varphi$ . Then, there exists a bounded DT  $\mathcal{T}'$  such that  $\mathcal{T}' \models \varphi$ .*

The proof makes use of some instruments and formalisms for *First-Order Logic* (FOL, for short) that are introduced in [MP15]. For the sake of completeness, here we give an informal discussion of such object. A *language signature* is a tuple  $\mathcal{L} = \langle \text{Ar}, \text{Rl}, \text{ar} \rangle$  in which  $\text{Ar}$  and  $\text{Rl}$  are two finite non-empty sets of *arguments* and *relations*, respectively, and  $\text{ar} : \text{Rl} \rightarrow 2^{\text{Ar}} \setminus \{\emptyset\}$  is a function mapping each relation in  $\text{Rl}$  to its non-empty set of arguments. Language signatures are used to reformulate FOL syntax in terms of *binding forms*, which are a way to associate variables to relations by means of bindings. The interpretation of FOL formulas is given on *relational structures*, which are tuples  $\mathcal{R} = \langle \text{Dm},$

<sup>12</sup>By  $\text{prj}_{\text{Pth}}$  we are denoting the natural lifting of the function  $\text{prj}_{\text{Trk}}$  to paths.

rl) with  $\text{Dm}$  being a non-empty domain and where  $\text{rl}(r) \subseteq \text{ar}(r) \rightarrow \text{Dm}$  is a set of functions, representing the tuples on which the relation  $r \in \text{Rl}$  is interpreted as true.

*Proof Idea.* The key idea used to prove the theorem is based on the *finite model property* of the *One-Binding* fragment of FOL (FOL[1B], for short), proved in [MP15], which allows to define a bounded-tree model  $\mathcal{T}'$ , which preserves the satisfiability of  $\varphi$ . In particular, for each state  $s^*$  of a tree  $\mathcal{T}$  satisfying  $\varphi$ , we build a first-order structure and a FOL[1B] formula  $\eta_{s^*}$  that characterizes the topology of the successors of  $s_*$  in  $\mathcal{T}$ . Then, since FOL[1B] enjoys the finite model property, we are able to build a finite first-order structure for  $\eta_{s^*}$  from which we can build the bounded model  $\mathcal{T}'$  of  $\varphi$ . Such a construction is based on both the disjoint and behavioral satisfiability of SL[1G]. For each state  $s^*$  in  $\mathcal{T}$ , we consider a set given by pairs of subsentences  $\eta$  of  $\varphi$  and states  $s$ , on which it holds that  $\mathcal{T}, s \models_{\theta} \eta$ , where such satisfaction is forced to pass through  $s^*$  for at least one universal assignment fed to  $\theta$ . This means that at least one play used to satisfy  $\eta$  passes through  $s^*$ . By the disjoint satisfiability, we have to cope with at most two **Sdfs** for each subsentence  $\eta$ , those given by  $\text{head}_{\eta}$  and  $\text{body}_{\eta}$ , implying that the total number of **Sdfs** to take into account, for all  $s^*$ , is finite. From that, we define a related FOL[1B] sentence  $\eta_{s^*}$ , having a model derived from the topology of the successors of  $s^*$  whose elements are constituted by the actions of  $\mathcal{T}$ . Now, by applying the finite model property to  $\eta_{s^*}$ , we derive the existence of a model for the formula  $\eta_{s^*}$  with a finite domain  $\text{Ac}'_{\eta, s^*}$ . Exploiting the finite model built for all states  $s^*$ , we are able to define the labeling of  $\mathcal{T}'$  and a behavioral **Sdf**  $\theta'$  in such a way that  $\mathcal{T}', \varepsilon \models_{\theta'} \varphi$ . □

*Proof of Theorem 5.4.* We give the proof for the case of  $\varphi = \wp \flat \psi$ , since the Boolean combination of principal sentences easily follows from this one. Given a tree-model  $\mathcal{T}$  for  $\varphi$ , derived by the tree-model property of SL[1G] of Theorem 4.13, for each state  $s^* \in \text{St}$ , consider the set  $\Phi_{s^*} \subseteq \text{St} \times \text{psnt}(\varphi)$  of states of  $\mathcal{T}$  and principal subsentences of  $\varphi$  such that  $(s, \eta) \in \Phi_{s^*}$  iff (i)  $\mathcal{T}, s \models \eta$  and (ii) there exists an assignment  $\chi \in \text{Asg}_{\text{Str}}(\llbracket \varphi_{\eta} \rrbracket)$  such that  $s^* = (\text{play}(\theta_{(s, \eta)}^{s^*}(\chi)) \circ \flat_{\eta}, s)_n$ , for some  $n \in \mathbb{N}$ , where the behavioral **Sdf**  $\theta_{(s, \eta)}^{s^*}$  is defined by means of its adjoint, which is in its turn built from the functions  $\text{head}_{\eta}$  and  $\text{body}_{\eta}$ , given by Theorem 5.3, applied on  $\eta = \wp \flat \psi_{\eta}$ . Observe that, for a fixed  $\eta$ , if  $s_1, s_2 \in \text{St}$ , with  $s_1 \neq s^*$  and  $s_2 \neq s^*$ ,  $\widehat{\theta_{(s, \eta)}^{s^*}}(\rho_{s_1}) = \text{head}_{\eta}(\rho'_{s_1} \cdot \rho_{s_1}) = \text{head}_{\eta}(\rho'_{s_2} \cdot \rho_{s_2}) = \widehat{\theta_{(s, \eta)}^{s^*}}(\rho_{s_2})$ , where  $\rho_{s_1}$  and  $\rho_{s_2}$  are the unique tracks ending in  $s^*$  and starting in  $s_1$  and  $s_2$ , respectively, while  $\rho'_{s_1}$  and  $\rho'_{s_2}$  are the unique tracks such that  $\rho'_{s_1} \cdot \rho_{s_1}$  and  $\rho'_{s_2} \cdot \rho_{s_2}$  start from  $\varepsilon$ . Now, for a given **Sdf** over Actions  $\tilde{\theta} \in \text{SF}_{\text{Ac}}(\wp)$  and a given state  $s \in \text{St}$ , define the set  $\text{Succ}_{\wp, \flat}(s) \triangleq \{s' \in \text{St} : \exists \mathbf{v} \in \text{Ac}^{\llbracket \wp \rrbracket}. \text{tr}(s, \mathbf{v}(\mathbf{v}) \circ \flat) = s'\}$ , where  $\mathbf{v} \in \text{Ac}^{\llbracket \wp \rrbracket} \rightarrow \text{Ac}^{\text{Vr}(\wp)}$  is a **Sdf** for  $\wp$  over actions. Intuitively, the set  $\text{Succ}_{\wp, \flat}(s)$  defines the set of states that can be reached in one step from  $s$  by prescribing the agents that are bound by  $\flat$  to an existential variable to move according to the **Sdf**  $\mathbf{v}$ .

At this point, consider the language signature  $\mathcal{L} = \langle \text{Ar}, \text{Rl}, \text{ar} \rangle = \langle \text{Ag}, \text{AP}, \text{ar} \rangle$  with  $\text{ar}(p) = \text{Ag}$ , for all  $p \in \text{AP}$ , where each atomic proposition is viewed as a relation having the agents as arguments and, so, the decisions as elements of its interpretation. Moreover, for all sets  $P \subseteq \text{AP}$ , let  $\text{mask}^P = \bigwedge_{p \in P} p \wedge \bigwedge_{q \in \text{AP} \setminus P} \neg q$  be the FOL[1B] formula asserting that only the relations in  $P$  hold. Finally, for all  $(s, \eta) \in \Phi_{s^*}$ , consider the FOL[1B] sentence

$\eta_s^* = \wp_\eta \flat_\eta \bigvee_{s \in \text{Succ}_{\theta_{(s,\eta)}^{s^*}}(\text{trk}(s^*), b_\eta)} \text{mask}^{\text{ap}(s)}$ <sup>13</sup>. Clearly, by definition, each  $\eta_s^*$  is satisfied by

the relational structure  $\mathcal{R}_{s^*} = \langle \text{Ac}, \text{rl}_{s^*} \rangle$  with  $\text{rl}_{s^*}(p) \triangleq \{\delta \in \text{Dc} : p \in \text{ap}(\text{tr}(s^*, \delta))\}$ , where a relation  $p$  is interpreted as true on all decisions that allow  $s^*$  to reach a state in which  $p$  holds. Indeed, for each partial valuation  $\mathbf{v} \in \text{Ac}^{\llbracket \wp_\eta \rrbracket}$ , it holds that either  $s' = \text{tr}(s, \text{head}_\eta(s)(\mathbf{v})) \in \text{Succ}_{\text{head}_\eta(s), b_\eta}(s)$  or  $s'' = \text{tr}(s, \text{body}_\eta(\text{trk}(s))(\mathbf{v})) \in \text{Succ}_{\text{body}_\eta(\text{trk}(s)), b_\eta}(s)$ , which implies that either  $\mathcal{R}_{s^*}, \text{head}_\eta(s)(\mathbf{v}) \models \text{mask}^{\text{ap}(s')}$  or  $\mathcal{R}_{s^*}, \text{body}_\eta(\text{trk}(s))(\mathbf{v}) \models \text{mask}^{\text{ap}(s')}$ . Hence, we have that  $\mathcal{R}_{s^*} \models \eta_s^*$  and, since this is true for all  $(s, \eta) \in \Phi_{s^*}$ , we derive that  $\mathcal{R}_{s^*} \models \bigwedge_{(s,\eta) \in \Phi_{s^*}} \eta_s^*$ .

At this point, from the finite model property of FOL[1B], we derive that there exists a finite relational structure  $\mathcal{R}'_{s^*} = \langle \text{Dm}'_{s^*}, \text{rl}'_{s^*} \rangle$  such that  $\mathcal{R}'_{s^*} \models \bigwedge_{(s,\eta) \in \Phi_{s^*}} \eta_s^*$ . Moreover, we define  $\widetilde{\theta'_{(s,\eta)}^{s^*}}$  to be such that  $\mathcal{R}'_{s^*} \models \widetilde{\theta'_{(s,\eta)}^{s^*}} \eta_s^*$ . Observe that, in the proof of finite model

property for FOL[1B] [MP15], the bound on  $\text{Dm}'_{s^*}$  only depends on the quantification and binding prefixes given in the formulas  $\eta_s^*$ , which all occur in  $\varphi$ . Thus, the size of  $\text{Dm}'_{s^*}$  does not depend on  $\eta$  and, *w.l.o.g.*, we can assume that  $\text{Dm}'_{s^*} = \text{Ac}'$  for all  $s^* \in \text{St}$ . Moreover, again from the finite model property proof of FOL[1B], there exists a function  $\text{M}_{s^*} : \text{Dc}' \rightarrow \text{Dc}$ , with  $\text{Dc}' = \text{Ac}'^{\text{Ag}}$ , such that, for all  $(s, \eta) \in \Phi_{s^*}$  and  $\delta' \in \text{rng}(\widetilde{\theta'_{(s,\eta)}^{s^*}})$ , we have that  $\text{M}_{s^*}(\delta') \in \text{rng}(\widetilde{\theta_{(s,\eta)}^{s^*}})$ , where  $\widetilde{\theta_{(s,\eta)}^{s^*}}$  is the  $\text{Sdf}$  used to satisfy  $\eta_s^*$  on  $\mathcal{R}$ . At this point, we define a DT  $\mathcal{T}'$  having  $\text{Ac}'$  as set of actions. In order to define the labeling function  $\text{ap}'$ , first consider the mapping  $\Gamma : \text{St}' \rightarrow \text{St}$  recursively defined as follows:

- $\Gamma(\varepsilon) = \varepsilon$ ;
- $\Gamma(s' \cdot \delta') = \Gamma(s') \cdot \text{M}_{\Gamma(s')}(\delta')$ .

By means of  $\Gamma$ , define  $\text{ap}'(s') \triangleq \text{ap}(\Gamma(s'))$  for all  $s' \in \text{St}'$ . It remains to prove that  $\mathcal{T}' \models \varphi$ . We do this by using a  $\text{Sdf}$   $\theta' \in \text{SF}_{\text{Str}'(\varphi)}$  defined from the adjoint  $\widetilde{\theta}'$  introduced in the following.

Let  $\rho'$  be a track in  $\mathcal{T}'$  and consider  $s' = \text{lst}(\rho')$ . If  $(\varepsilon, \varphi) \in \Phi_{\Gamma(s')}$ , define  $\widetilde{\theta}'(\rho') = \theta'_{(s,\eta)}^{s^*}$ . For all other tracks, the value of  $\widetilde{\theta}'$  may be arbitrary. We now show that  $\mathcal{T}, \emptyset, \varepsilon \models_{\theta'} \varphi$ , by induction on the nesting of principal subsentences. As base case, suppose that  $\varphi$  has nesting 0. This implies that it is of the form  $\wp b \psi$  with  $\psi$  being an LTL formula. Then, consider a universal assignment  $\chi' \in \text{Asg}_{\text{Str}'(\llbracket \varphi \rrbracket)}$  and then the assignment  $\theta'(\chi')$ . This determines a play  $\pi' = \text{play}(\theta'(\chi'), \varepsilon)$ . Now, consider a universal assignment  $\chi \in \text{Asg}_{\text{Str}(\llbracket \varphi \rrbracket)}$  such that, for all  $x \in \llbracket \varphi \rrbracket$  and  $\rho' \in \text{Trk}'(\varepsilon)$ , it holds that  $\chi'(x)(\rho') = \chi(x)(\Gamma(\rho'))$ , where  $\Gamma$  is the lifting over tracks of the mapping over states defined above, *i.e.*, by  $\Gamma(\rho')$  is the track in  $\mathcal{T}$  obtained from  $\rho'$  by mapping each state  $s'$  in  $\rho'$  into  $s = \Gamma(s')$ . It holds that  $\pi = \text{play}(\theta(\chi) \circ b, \varepsilon)$  is such that, for all  $i \in \mathbb{N}$ ,  $(\pi)_{\leq i} = \Gamma((\pi')_{\leq i})$ . Indeed, by induction on  $i$ , as base case, we have that  $(\pi)_{\leq 0} = \varepsilon = \Gamma(\varepsilon) = (\pi')_{\leq 0}$ . As inductive case, suppose that  $(\pi)_{\leq i} = \Gamma((\pi')_{\leq i})$ . Then,  $(\pi)_{i+1} = \text{tr}((\pi)_i, \widetilde{\theta}((\pi)_{\leq i}) \circ b) = \text{tr}(\Gamma((\pi')_i), \widetilde{\theta}(\Gamma((\pi')_{\leq i})) \circ b) = \Gamma(\text{tr}((\pi')_i, \widetilde{\theta}'((\pi')_{\leq i}) \circ b)) = \Gamma((\pi')_{i+1})$ . Thus, according to the definition of  $\text{ap}'$ , we have that  $\text{ap}'((\pi')_i) = \text{ap}((\pi)_i)$ , for all  $i \in \mathbb{N}$ . Since  $\pi \models \psi$ , we derive that  $\pi' \models \psi$ . This holds for all possible universal assignments  $\chi' \in \text{Asg}_{\text{Str}'(\llbracket \varphi \rrbracket)}$ . Hence, it holds that  $\mathcal{T}' \models_{\theta'} \wp b \psi$ . The inductive case follows by considering the principal subsentences of  $\varphi$  as fresh atomic propositions.  $\square$

<sup>13</sup>By  $\text{trk}(s)$  we are denoting the unique track starting from  $\varepsilon$  and ending in  $s$ .

**5.2. Alternating tree automata.** *Nondeterministic tree automata* are a generalization to infinite trees of the classical *nondeterministic word automata* on infinite words (see [Tho90], for an introduction). *Alternating tree automata* are a further generalization of nondeterministic tree automata [MS87]. Intuitively, on visiting a node of the input tree, while the latter sends exactly one copy of itself to each of the successors of the node, the former can send several own copies to the same successor. Here we use, in particular, *alternating parity tree automata*, which are alternating tree automata along with a *parity acceptance condition* (see [GTW02], for a survey).

We now give the formal definition of alternating tree automata.

**Definition 5.5** (Alternating Tree Automata). An *alternating tree automaton* (ATA, for short) is a tuple  $\mathcal{A} \triangleq \langle \Sigma, \text{Dir}, \mathbb{Q}, \delta, q_0, \aleph \rangle$ , where  $\Sigma$ ,  $\text{Dir}$ , and  $\mathbb{Q}$  are, respectively, non-empty finite sets of *input symbols*, *directions*, and *states*,  $q_0 \in \mathbb{Q}$  is an *initial state*,  $\aleph$  is an *acceptance condition* to be defined later, and  $\delta : \mathbb{Q} \times \Sigma \rightarrow \mathcal{B}^+(\text{Dir} \times \mathbb{Q})$  is an *alternating transition function* that maps each pair of states and input symbols to a positive Boolean combination on the set of propositions of the form  $(d, q) \in \text{Dir} \times \mathbb{Q}$ , a.k.a. *moves*.

On one hand, a *nondeterministic tree automaton* (NTA, for short) is a special case of ATA in which each conjunction in the transition function  $\delta$  has exactly one move  $(d, q)$  associated with each direction  $d$ . This means that, for all states  $q \in \mathbb{Q}$  and symbols  $\sigma \in \Sigma$ , we have that  $\delta(q, \sigma)$  is equivalent to a Boolean formula of the form  $\bigvee_i \bigwedge_{d \in \text{Dir}} (d, q_{i,d})$ . On the other hand, a *universal tree automaton* (UTA, for short) is a special case of ATA in which all the Boolean combinations that appear in  $\delta$  are conjunctions of moves. Thus, we have that  $\delta(q, \sigma) = \bigwedge_i (d_i, q_i)$ , for all states  $q \in \mathbb{Q}$  and symbols  $\sigma \in \Sigma$ .

The semantics of the ATAs is given through the following concept of run.

**Definition 5.6** (ATA Run). A *run* of an ATA  $\mathcal{A} = \langle \Sigma, \text{Dir}, \mathbb{Q}, \delta, q_0, \aleph \rangle$  on a  $\Sigma$ -labeled Dir-tree  $\mathcal{T} = \langle \text{T}, \nu \rangle$  is a  $(\text{Dir} \times \mathbb{Q})$ -tree  $\text{R}$  such that, for all nodes  $x \in \text{R}$ , where  $x = \prod_{i=1}^n (d_i, q_i)$  and  $y \triangleq \prod_{i=1}^n d_i$  with  $n \in [0, \omega[$ , it holds that (i)  $y \in \text{T}$  and (ii), there is a set of moves  $S \subseteq \text{Dir} \times \mathbb{Q}$  with  $S \models \delta(q_n, \nu(y))$  such that  $x \cdot (d, q) \in \text{R}$ , for all  $(d, q) \in S$ .

In the following, we consider ATAs along with the *parity acceptance condition* (APT, for short)  $\aleph \triangleq (F_1, \dots, F_k) \in (2^{\mathbb{Q}})^+$  with  $F_1 \subseteq \dots \subseteq F_k = \mathbb{Q}$  (see [KVV00], for more). The number  $k$  of sets in the tuple  $\aleph$  is called the *index* of the automaton. We also consider ATAs with the *co-Büchi acceptance condition* (ACT, for short) that is the special parity condition with index 2.

Let  $\text{R}$  be a run of an ATA  $\mathcal{A}$  on a tree  $\mathcal{T}$  and  $w$  one of its branches. Then, by  $\text{inf}(w) \triangleq \{q \in \mathbb{Q} : |\{i \in \mathbb{N} : \exists d \in \text{Dir}. (w)_i = (d, q)\}| = \omega\}$  we denote the set of states that occur infinitely often as the second component of the letters along the branch  $w$ . Moreover, we say that  $w$  satisfies the parity acceptance condition  $\aleph = (F_1, \dots, F_k)$  if the least index  $i \in [1, k]$  for which  $\text{inf}(w) \cap F_i \neq \emptyset$  is even.

At this point, we can define the concept of language accepted by an ATA.

**Definition 5.7** (ATA Acceptance). An ATA  $\mathcal{A} = \langle \Sigma, \text{Dir}, \mathbb{Q}, \delta, q_0, \aleph \rangle$  *accepts* a  $\Sigma$ -labeled Dir-tree  $\mathcal{T}$  iff there exists a run  $\text{R}$  of  $\mathcal{A}$  on  $\mathcal{T}$  such that all its infinite branches satisfy the acceptance condition  $\aleph$ .

By  $L(\mathcal{A})$  we denote the language accepted by the ATA  $\mathcal{A}$ , i.e., the set of trees  $\mathcal{T}$  accepted by  $\mathcal{A}$ . Moreover,  $\mathcal{A}$  is said to be *empty* if  $L(\mathcal{A}) = \emptyset$ . The *emptiness problem* for  $\mathcal{A}$  is to decide whether  $L(\mathcal{A}) = \emptyset$ .

**5.3. Satisfiability procedure.** We finally solve the satisfiability problem for SL[1G] and show that it is 2EXPTIME-COMplete, as for ATL<sup>\*</sup>. The algorithmic procedure is based on an automata-theoretic approach, which reduces the decision problem for the logic to the emptiness problem of a suitable universal Co-Büchi tree automaton (UCT, for short) [GTW02]. From an high-level point of view, the automaton construction seems similar to what was proposed in literature for CTL<sup>\*</sup> [KVV00] and ATL<sup>\*</sup> [Sch08]. However, our technique is completely new, since it is based on the novel notions of behavioral semantics and disjoint satisfiability.

**Principal sentences.** To proceed, we first have to introduce the concept of encoding for an assignment and the labeling of a DT.

**Definition 5.8** (Assignment-Labeling Encoding). Let  $\mathcal{T}$  be a DT,  $t \in \text{St}_{\mathcal{T}}$  one of its states, and  $\chi \in \text{Asg}_{\mathcal{T}}(V, t)$  an assignment defined on the set  $V \subseteq \text{Vr}$ . A  $(\text{Val}_{\text{Ac}_{\mathcal{T}}}(V) \times 2^{\text{AP}})$ -labeled Dc $_{\mathcal{T}}$ -tree  $\mathcal{T}' \triangleq \langle \text{St}_{\mathcal{T}}, u \rangle$  is an *assignment-labeling encoding* for  $\chi$  on  $\mathcal{T}$  if  $u(\text{lst}((\rho)_{\geq 1})) = (\widehat{\chi}(\rho), \text{ap}_{\mathcal{T}}(\text{lst}(\rho)))$ , for all  $\rho \in \text{Trk}_{\mathcal{T}}(t)$ <sup>14</sup>.

Observe that there is a unique assignment-labeling encoding for each assignment over a given DT.

Now, we prove the existence of a UCT  $\mathcal{U}_{b\psi}^{\text{Ac}}$  for each SL[1G] goal  $b\psi$  having no principal subsentences. The  $\mathcal{U}_{b\psi}^{\text{Ac}}$  recognizes all the assignment-labeling encodings  $\mathcal{T}'$  of an a priori given assignment  $\chi$  over a generic DT  $\mathcal{T}$ , whenever the goal is satisfied on  $\mathcal{T}$  under  $\chi$ . Intuitively, we start with a UCW, recognizing all infinite words on the alphabet  $2^{\text{AP}}$  that satisfy the LTL formula  $\psi$ , obtained by a simple variation of the Vardi-Wolper construction [VW86a]. Then, we run it on the encoding tree  $\mathcal{T}'$  by following the directions identified by the assignment in its labeling.

**Lemma 5.9** (SL[1G] Goal Automaton). *Let  $b\psi$  an SL[1G] goal without principal subsentences and Ac a finite set of actions. Then, there exists an UCT  $\mathcal{U}_{b\psi}^{\text{Ac}} \triangleq \langle \text{Val}_{\text{Ac}}(\text{free}(b\psi)) \times 2^{\text{AP}}, \text{Dc}, \mathcal{Q}_{b\psi}, \delta_{b\psi}, q_{0b\psi}, \aleph_{b\psi} \rangle$  such that, for all DT  $s \mathcal{T}$  with  $\text{Ac}_{\mathcal{T}} = \text{Ac}$ , states  $t \in \text{St}_{\mathcal{T}}$ , and  $t$ -total assignments  $\chi \in \text{Asg}_{\mathcal{T}}(\text{free}(b\psi), t)$ , it holds that  $\mathcal{T}, \chi, t \models b\psi$  iff  $\mathcal{T}' \in \text{L}(\mathcal{U}_{b\psi}^{\text{Ac}})$ , where  $\mathcal{T}'$  is the assignment-labeling encoding for  $\chi$  on  $\mathcal{T}$ .*

*Proof.* A first step in the construction of the UCT  $\mathcal{U}_{b\psi}^{\text{Ac}}$ , is to consider the UCW  $\mathcal{U}_{\psi} \triangleq \langle 2^{\text{AP}}, \mathcal{Q}_{\psi}, \delta_{\psi}, \mathcal{Q}_{0\psi}, \aleph_{\psi} \rangle$  obtained by dualizing the NBW resulting from the application of the classic Vardi-Wolper construction to the LTL formula  $\neg\psi$  [VW86a]. Observe that  $\text{L}(\mathcal{U}_{\psi}) = \text{L}(\psi)$ , i.e., this automaton recognizes all infinite words on the alphabet  $2^{\text{AP}}$  that satisfy the LTL formula  $\psi$ . Then, define the components of  $\mathcal{U}_{b\psi}^{\text{Ac}} \triangleq \langle \text{Val}_{\text{Ac}}(\text{free}(b\psi)) \times 2^{\text{AP}}, \text{Dc}, \mathcal{Q}_{b\psi}, \delta_{b\psi}, q_{0b\psi}, \aleph_{b\psi} \rangle$ , as follows:

- $\mathcal{Q}_{b\psi} \triangleq \{q_{0b\psi}\} \cup \mathcal{Q}_{\psi}$ , with  $q_{0b\psi} \notin \mathcal{Q}_{\psi}$ ;
- $\delta_{b\psi}(q_{0b\psi}, (v, \sigma)) \triangleq \bigwedge_{q \in \mathcal{Q}_{0\psi}} \delta_{b\psi}(q, (v, \sigma))$ , for all  $(v, \sigma) \in \text{Val}_{\text{Ac}}(\text{free}(b\psi)) \times 2^{\text{AP}}$ ;
- $\delta_{b\psi}(q, (v, \sigma)) \triangleq \bigwedge_{q' \in \delta_{\psi}(q, \sigma)} (v \circ b, q')$ , for all  $q \in \mathcal{Q}_{\psi}$  and  $(v, \sigma) \in \text{Val}_{\text{Ac}}(\text{free}(b\psi)) \times 2^{\text{AP}}$ ;
- $\aleph_{b\psi} \triangleq \aleph_{\psi}$ .

Intuitively, the UCT  $\mathcal{U}_{b\psi}^{\text{Ac}}$  simply runs the UCW  $\mathcal{U}_{\psi}$  on the branch of the encoding individuated by the assignment in input. Thus, it is easy to see that, for all states  $t \in \text{St}_{\mathcal{T}}$  and

<sup>14</sup>Note that  $\text{lst}(\varepsilon) = \varepsilon$ .

$t$ -total assignments  $\chi \in \text{Asg}_{\mathcal{T}}(\text{free}(b\psi), t)$ , it holds that  $\mathcal{T}, \chi, t \models b\psi$  iff  $\mathcal{T}' \in L(\mathcal{U}_{b\psi}^{\text{Ac}})$ , where  $\mathcal{T}'$  is the assignment-labeling encoding for  $\chi$  on  $\mathcal{T}$ .  $\square$

We now introduce a new concept of encoding regarding the behavioral dependence maps over strategies.

**Definition 5.10** (Behavioral Dependence-Labeling Encoding). Let  $\mathcal{T}$  be a DT,  $t \in \text{St}_{\mathcal{T}}$  one of its states, and  $\theta \in \text{BSF}_{\text{Str}_{\mathcal{T}}}(\wp)$  a behavioral dependence map over strategies for a quantification prefix  $\wp \in \text{Qnt}(\text{V})$  over the set  $\text{V} \subseteq \text{Vr}$ . A  $(\text{SF}_{\text{Ac}_{\mathcal{T}}}(\wp) \times 2^{\text{AP}})$ -labeled Dir-tree  $\mathcal{T}' \triangleq \langle \text{St}_{\mathcal{T}}, \mathbf{u} \rangle$  is a *behavioral dependence-labeling encoding* for  $\theta$  on  $\mathcal{T}$  if  $\mathbf{u}(\text{lst}((\rho)_{\geq 1})) = (\tilde{\theta}(\rho), \text{ap}_{\mathcal{T}}(\text{lst}(\rho)))$ , for all  $\rho \in \text{Trk}_{\mathcal{T}}(t)$ .

Observe that also in this case there exists a unique behavioral dependence-labeling encoding for each behavioral dependence map over strategies.

Finally, in the next lemma, we show how to locally handle the strategy quantifications on each state of the model, by simply using a quantification over actions modeled by the choice of an action dependence map. Intuitively, we guess in the labeling what is the right part of the dependence map over strategies for each node of the tree and then verify that, for all assignments of universal variables, the corresponding complete assignment satisfies the inner formula.

**Lemma 5.11** (SL[1G] Sentence Automaton). *Let  $\wp b\psi$  be an SL[1G] principal sentence without principal subsentences and  $\text{Ac}$  a finite set of actions. Then, there exists an UCT  $\mathcal{U}_{\wp b\psi}^{\text{Ac}} \triangleq \langle \text{SF}_{\text{Ac}}(\wp) \times 2^{\text{AP}}, \text{Dc}, \mathbf{Q}_{\wp b\psi}, \delta_{\wp b\psi}, q_{0\wp b\psi}, \aleph_{\wp b\psi} \rangle$  such that, for all DT  $s \mathcal{T}$  with  $\text{Ac}_{\mathcal{T}} = \text{Ac}$ , states  $t \in \text{St}_{\mathcal{T}}$ , and behavioral dependence maps over strategies  $\theta \in \text{BSF}_{\text{Str}_{\mathcal{T}}}(\wp)$ , it holds that  $\mathcal{T}, \theta(\chi), t \models_{\text{B}} b\psi$ , for all  $\chi \in \text{Asg}_{\mathcal{T}}(\llbracket \wp \rrbracket, t)$ , iff  $\mathcal{T}' \in L(\mathcal{U}_{b\psi}^{\text{Ac}})$ , where  $\mathcal{T}'$  is the behavioral dependence-labeling encoding for  $\theta$  on  $\mathcal{T}$ .*

*Proof.* By Lemma 5.9 of SL[1G] goal automaton, there is an UCT  $\mathcal{U}_{b\psi}^{\text{Ac}} \triangleq \langle \text{Val}_{\text{Ac}}(\text{free}(b\psi)) \times 2^{\text{AP}}, \text{Dc}, \mathbf{Q}_{b\psi}, \delta_{b\psi}, q_{0b\psi}, \aleph_{b\psi} \rangle$  such that, for all DTs  $\mathcal{T}$  with  $\text{Ac}_{\mathcal{T}} = \text{Ac}$ , states  $t \in \text{St}_{\mathcal{T}}$ , and assignments  $\chi \in \text{Asg}_{\mathcal{T}}(\text{free}(b\psi), t)$ , it holds that  $\mathcal{T}, \chi, t \models b\psi$  iff  $\mathcal{T}' \in L(\mathcal{U}_{b\psi}^{\text{Ac}})$ , where  $\mathcal{T}'$  is the assignment-labeling encoding for  $\chi$  on  $\mathcal{T}$ .

Now, transform  $\mathcal{U}_{b\psi}^{\text{Ac}}$  into the new UCT  $\mathcal{U}_{\wp b\psi}^{\text{Ac}} \triangleq \langle \text{SF}_{\text{Ac}}(\wp) \times 2^{\text{AP}}, \text{Dc}, \mathbf{Q}_{\wp b\psi}, \delta_{\wp b\psi}, q_{0\wp b\psi}, \aleph_{\wp b\psi} \rangle$ , with  $\mathbf{Q}_{\wp b\psi} \triangleq \mathbf{Q}_{b\psi}$ ,  $q_{0\wp b\psi} \triangleq q_{0b\psi}$ , and  $\aleph_{\wp b\psi} \triangleq \aleph_{b\psi}$ , which is used to handle the quantification prefix  $\wp$  atomically, where the transition function is defined as follows:  $\delta_{\wp b\psi}(q, (\theta, \sigma)) \triangleq \bigwedge_{\mathbf{v} \in \text{Val}_{\text{Ac}}(\llbracket \wp \rrbracket)} \delta_{b\psi}(q, (\theta(\mathbf{v}), \sigma))$ , for all  $q \in \mathbf{Q}_{\wp b\psi}$  and  $(\theta, \sigma) \in \text{SF}_{\text{Ac}}(\wp) \times 2^{\text{AP}}$ . Intuitively,  $\mathcal{U}_{\wp b\psi}^{\text{Ac}}$  reads an action dependence map  $\theta$  on each node of the input tree  $\mathcal{T}'$  labeled with a set of atomic propositions  $\sigma$  and simulates the execution of the transition function  $\delta_{b\psi}(q, (\mathbf{v}, \sigma))$  of  $\mathcal{U}_{b\psi}^{\text{Ac}}$ , for each possible valuation  $\mathbf{v} = \theta(\mathbf{v}')$  on  $\text{free}(b\psi)$  obtained from  $\theta$  via a universal valuation  $\mathbf{v}' \in \text{Val}_{\text{Ac}}(\llbracket \wp \rrbracket)$ . It is worth observing that we cannot move the component set  $\text{SF}_{\text{Ac}}(\wp)$  from the input alphabet to the states of  $\mathcal{U}_{\wp b\psi}^{\text{Ac}}$  by making a related guessing of the dependence map  $\theta$  in the transition function, since the automaton is universal and we have to ensure that all states in a given node of the tree  $\mathcal{T}'$ , *i.e.*, in each track of the original model  $\mathcal{T}$ , make the same choice for  $\theta$ .

Finally, it remains to prove that, for all states  $t \in \text{St}_{\mathcal{T}}$  and behavioral dependence maps over strategies  $\theta \in \text{BSF}_{\text{Str}_{\mathcal{T}t}}(\wp)$ , it holds that  $\mathcal{T}, \theta(\chi), t \models_{\text{B}} b\psi$ , for all  $\chi \in \text{Asg}_{\mathcal{T}}(\llbracket \wp \rrbracket, t)$ , iff  $\mathcal{T}' \in L(\mathcal{U}_{\wp b\psi}^{\text{Ac}})$ , where  $\mathcal{T}'$  is the behavioral dependence-labeling encoding for  $\theta$  on  $\mathcal{T}$ .

[*Only if*]. Suppose that  $\mathcal{T}, \theta(\chi), t \models_{\text{B}} b\psi$ , for all  $\chi \in \text{Asg}_{\mathcal{T}}(\llbracket \varphi \rrbracket, t)$ . Since  $\psi$  does not contain principal subsentences, we have that  $\mathcal{T}, \theta(\chi), t \models b\psi$ . So, due to the property of  $\mathcal{U}_{b\psi}^{\text{Ac}}$ , it follows that there exists an assignment-labeling encoding  $\mathcal{T}'_{\chi} \in \text{L}(\mathcal{U}_{b\psi}^{\text{Ac}})$ , which implies the existence of a  $(\text{Dc} \times \text{Q}_{b\psi})$ -tree  $\text{R}_{\chi}$  that is an accepting run for  $\mathcal{U}_{b\psi}^{\text{Ac}}$  on  $\mathcal{T}'_{\chi}$ . At this point, let  $\text{R} \triangleq \bigcup_{\chi \in \text{Asg}_{\mathcal{T}}(\llbracket \varphi \rrbracket, t)} \text{R}_{\chi}$  be the union of all runs. Then, due to the particular definition of the transition function of  $\mathcal{U}_{\wp b\psi}^{\text{Ac}}$ , it is not hard to see that  $\text{R}$  is an accepting run for  $\mathcal{U}_{\wp b\psi}^{\text{Ac}}$  on  $\mathcal{T}'$  defined as above. Hence,  $\mathcal{T}' \in \text{L}(\mathcal{U}_{\wp b\psi}^{\text{Ac}})$ .

[*If*]. Suppose that  $\mathcal{T}' \in \text{L}(\mathcal{U}_{\wp b\psi}^{\text{Ac}})$ . Then, there exists a  $(\text{Dc} \times \text{Q}_{\wp b\psi})$ -tree  $\text{R}$  that is an accepting run for  $\mathcal{U}_{\wp b\psi}^{\text{Ac}}$  on  $\mathcal{T}'$ . Now, for each  $\chi \in \text{Asg}_{\mathcal{T}}(\llbracket \varphi \rrbracket, t)$ , let  $\text{R}_{\chi}$  be the run for  $\mathcal{U}_{b\psi}^{\text{Ac}}$  on the assignment-state encoding  $\mathcal{T}'_{\chi}$  for  $\theta(\chi)$  on  $\mathcal{T}$ . Due to the particular definition of the transition function of  $\mathcal{U}_{\wp b\psi}^{\text{Ac}}$ , it is not hard to see that  $\text{R}_{\chi} \subseteq \text{R}$ . Thus, since  $\text{R}$  is accepting, we have that  $\text{R}_{\chi}$  is accepting as well. So,  $\mathcal{T}'_{\chi} \in \text{L}(\mathcal{U}_{b\psi}^{\text{Ac}})$ . At this point, due to the property of  $\mathcal{U}_{b\psi}^{\text{Ac}}$ , it follows that  $\mathcal{T}, \theta(\chi), t \models b\psi$ . Since  $\psi$  does not contain principal subsentences, we have that  $\mathcal{T}, \theta(\chi), t \models_{\text{B}} b\psi$ , for all  $\chi \in \text{Asg}_{\mathcal{T}}(\llbracket \varphi \rrbracket, t)$ .  $\square$

**Full sentences.** By summing up all previous results, we are now able to solve the satisfiability problem for the full  $\text{SL}[1\text{G}]$  fragment.

To construct the automaton for a given  $\text{SL}[1\text{G}]$  sentence  $\varphi$ , we first consider all  $\text{UCT } \mathcal{U}_{\phi}^{\text{Ac}}$ , for an assigned bounded set  $\text{Ac}$ , previously described for the principal sentences  $\phi \in \text{psnt}(\varphi)$ , in which the inner subsentences are considered as atomic propositions. Then, thanks to the disjoint satisfiability property of Definition 5.1, we can merge them into a unique  $\text{UCT } \mathcal{U}_{\varphi}$  that supplies the dependence map labeling of internal components  $\mathcal{U}_{\phi}^{\text{Ac}}$ , by using the two functions **head** and **body** contained into its labeling. Moreover, observe that the final automaton runs on a  $b$ -bounded decision tree, where  $b$  is obtained from Theorem 5.4 on the bounded-tree model property.

**Theorem 5.12** ( $\text{SL}[1\text{G}]$  Automaton). *Let  $\varphi$  be an  $\text{SL}[1\text{G}]$  sentence. Then, there exists an  $\text{UCT } \mathcal{U}_{\varphi}$  such that  $\varphi$  is satisfiable iff  $\text{L}(\mathcal{U}_{\varphi}) \neq \emptyset$ .*

Finally, by a simple calculation of the size of  $\mathcal{U}_{\varphi}$  and the complexity of the related emptiness problem, we state in the next theorem the precise computational complexity of the satisfiability problem for  $\text{SL}[1\text{G}]$ .

**Theorem 5.13** ( $\text{SL}[1\text{G}]$  Satisfiability). *The satisfiability problem for  $\text{SL}[1\text{G}]$  is  $2\text{EXPTIME-COMplete}$ .*

*Proof.* By Theorem 5.12 of  $\text{SL}[1\text{G}]$  automaton, to verify whether an  $\text{SL}[1\text{G}]$  sentence  $\varphi$  is satisfiable we can calculate the emptiness of the  $\text{UPT } \mathcal{U}_{\varphi}$ . This automaton is obtained by merging all  $\text{UCTs } \mathcal{U}_{\phi}^{\text{Ac}}$ , with  $\phi = \wp b\psi \in \text{psnt}(\varphi)$ , which in turn are based on the  $\text{UCTs } \mathcal{U}_{b\psi}^{\text{Ac}}$  that embed the  $\text{UCWs } \mathcal{U}_{\psi}$ . By a simple calculation, it is easy to see that  $\mathcal{U}_{\varphi}$  has  $2^{O(|\varphi|)}$  states. Indeed, by the Vardi-Wolper construction, all the  $\text{UCWs } \mathcal{U}_{\psi}$  are of size bounded by  $2^{|\psi|}$ . Consequently, due to Lemma 5.9, also the  $\text{UCWs } \mathcal{U}_{b\psi}$  have the same bound on the state space. Therefore, due to the construction of Lemma 5.11, the cardinality of the state space of the  $\text{UCWs } \mathcal{U}_{\wp b\psi}$  is  $O(2^{|\wp b\psi|})$ . Finally, since all the  $\wp b\psi$  occur into  $\varphi$ , we obtain that the size of the  $\text{UCW } \mathcal{U}_{\varphi}$  is bounded by  $2^{|\varphi|}$ .

Now, by using a well-known nondeterminization procedure for  $\text{APTs}$  [MS95], we obtain an equivalent  $\text{NPT } \mathcal{N}_{\varphi}$  with  $2^{2^{O(|\varphi|)}}$  states and index  $2^{O(|\varphi|)}$ .

The emptiness problem for such a kind of automaton with  $n$  states and index  $h$  is solvable in time  $O(n^h)$ . Thus, we get that the time complexity of checking whether  $\varphi$  is satisfiable is  $2^{2^{O(|\varphi|)}}$ . Hence, the membership of the satisfiability problem for  $SL[1G]$  in  $2EXPTIME$  directly follows. Finally the thesis is proved, by getting the relative lower bound from the same problem for  $CTL^*$  [VS85].  $\square$

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